FLEXURAL RIGIDITY OF A RECTANGULAR STRIP OF SANDWICH CONSTRUCTION

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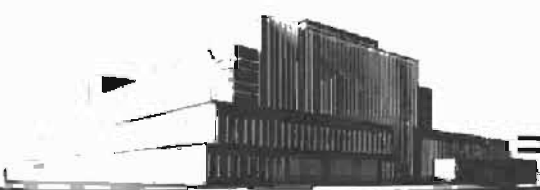

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UNITED STATES DEPARTMENT OF AGRICULTURE
FOREST SERVICE

In Cooperation with the University of Wisconsin

FLEXURAL RIGIDITY OF A RECTANGULAR STRIP

OF SANDWICH CONSTRUCTION[1]

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U.S. Department of Agriculture

The flexural rigidity of a sheet of sandwich is not so high as the usual calculations based on the Young's moduli of core and facings would indicate. The relatively low modulus of rigidity of the core causes a considerable reduction in the calculated stiffness, especially for sheets whose linear dimensions are not extremely large in comparison to their thickness. This reduction in stiffness is well known in the case of beams of small span-depth ratio.

A method[3] for calculating the effective stiffness of strips of plywood has been developed and applied to certain types of ordinary plywood. This method is used to obtain a formula for the calculation of the effective stiffness of a strip of sandwich construction supported at its ends and loaded at its center. The strip is considered to be made up of two cantilever beams. The relations between stress and strain and the conditions of equilibrium and strain compatibility in the facings and core of the sandwich strip lead to a differential equation that is satisfied by a stress function. A suitable stress function is chosen and fitted to the proper boundary conditions of each facing and of the core. When this is done it is found that only three constants remain to be determined by the conditions at the fixed end of the cantilever. These constants are determined by placing the horizontal displacements at the top surface of the

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[2] Maintained at Madison, Wis., in cooperation with the University of Wisconsin.

[3] March, H. W. Bending of a Centrally Loaded Rectangular Strip of Plywood. Physics, Vol. 7, 1936, pp. 32-41.

upper facing and at the bottom surface of the lower facing and the vertical displacement near the center of the core (at the origin of the coordinate system used) equal to zero. Thus the facings and the core are not restrained from rotating about their associated points of restraint except by their interactions with each other. The result is that their individual stiffnesses in bending are neglected at points directly under the central load and, therefore, that the theory developed leads to a conservative estimate of flexural rigidity if the individual stiffnesses of the facings contribute substantially to the total stiffness of the sandwich strip. Both core and faces will be assumed to be made of orthotropic material, such as wood. The result can be extended immediately to cases where one or both of the materials are isotropic.

The thickness of the facings will be denoted by f_1 and f_2, respectively, that of the core by c, and the total thickness by h. The width of the strip will be denoted by b. The neutral plane, $z = o$ in figure 1, is taken to be at a distance q from the facing whose thickness is f_1. The value of q will be determined in the course of the analysis. The difference, $c - q$, will be denoted by p.

The reduction in stiffness of a rectangular strip of length a, as shown in figure 1, will be determined by assuming a load P to be applied at the center along a line perpendicular to the direction of the span. The strip will be considered to be made up of two cantilevers fixed at their junction $x = 0$ and under the action of a load $\frac{P}{2}$ at the end of each, namely at $x = \frac{a}{2}$ and $x = -\frac{a}{2}$. The width of the strip will be taken to be large in comparison with its thickness, so that the cantilever may be considered to be approximately in a state of plane strain.

One of the cantilevers under consideration is shown in figure 2.

In the state of plane strain it is assumed that the components of displacement u and w parallel to the axes of x and z, respectively, are functions of x and z only, and that the component v parallel to the axis of y is zero. All components of stress and strain are independent of y. The strain components e_{xy}, e_{yz}, and e_{yy}, and the stress components X_y, Y_z all vanish. The stress components Y_y is, in general, not zero. Hence, to maintain the strip in a state of plane strain, tensile or compressive forces must be applied on the faces $y = 0$ and $y = b$ of the strip. The influence of these applied forces on the deflection of the cantilever is assumed to be negligible.

At the planes of separation between the facings and the core the following conditions hold:

The components of stress Z_z and X_z are continuous.

The components of displacement u and w are continuous.

Within each layer the components of strain and stress are connected by the following relations,[4] if the axes of x, y, and z are assumed to be normal to the planes of symmetry of the orthotropic materials of the faces and core.

$$
\left.
\begin{aligned}
e_{xx} &= \frac{1}{E_x} X_x - \frac{\sigma_{yx}}{E_y} Y_y - \frac{\sigma_{zx}}{E_z} Z_z, \\
e_{yy} &= - \frac{\sigma_{xy}}{E_x} X_x + \frac{1}{E_y} Y_y - \frac{\sigma_{zy}}{E_z} Z_z, \\
e_{zz} &= - \frac{\sigma_{xz}}{E_x} X_x - \frac{\sigma_{yz}}{E_y} Y_y + \frac{1}{E_z} Z_z,
\end{aligned}
\right\} \qquad (1)
$$

$$
E_{xz} = \frac{1}{\mu_{xz}} X_z \qquad (2)
$$

In these equations E_x, E_y, and E_z are Young's moduli in the directions x, y, and z, respectively. Poisson's ratio σ_{xy} is the ratio of the contraction parallel to the y-axis to the extension parallel to the x-axis associated with a tension parallel to the x-direction. The quantity μ_{xz} is the modulus of rigidity associated with the directions x and z.

In the respective layers the components of stress and strain and the constants of the materials will be denoted by the subscripts 1, 2, and c. The subscript 1 will refer to the facing of thickness f_1, 2 to the facing of thickness f_2, and c to the core.

Since

$$
e_{yy} = 0 \qquad (3)
$$

$$
Y_y = \frac{E_y}{E_x} \sigma_{xy} X_x + \frac{E_y}{E_z} \sigma_{zy} Z_z. \qquad (4)
$$

Substituting (4) in (1), it is found that in each layer

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[4] U. S. Forest Products Laboratory Report No. 1312, page 34. Love's notations for stress and strain components are used. See A. E. H. Love, "The Mathematical Theory of Elasticity."

$$e_{xx} = \frac{1}{E_x}(1 - \sigma_{xy}\,\sigma_{yx})\,X_x - \frac{1}{E_z}(\sigma_{yx}\,\sigma_{zy} + \sigma_{zx})\,Z_z$$

$$e_{zz} = -\frac{1}{E_x}(\sigma_{xy}\,\sigma_{yz} + \sigma_{xz})\,X_x + \frac{1}{E_z}(1 - \sigma_{yz}\,\sigma_{zy})\,Z_z \qquad (5)$$

Noting that[5]

$$\sigma_{yx} = \frac{E_y\,\sigma_{xy}}{E_x}, \quad \sigma_{zy} = \frac{E_z\,\sigma_{yz}}{E_y}, \quad \sigma_{zx} = \frac{E_z\,\sigma_{xz}}{E_x},$$

equations (5) may be written

$$e_{xx} = \alpha X_x - \beta Z_z,$$

$$e_{zz} = -\beta X_x + \gamma Z_z, \qquad (6)$$

where

$$\alpha = \frac{1}{E_x}(1 - \sigma_{xy}\,\sigma_{yx}), \quad \beta = \frac{1}{E_x}(\sigma_{xy}\,\sigma_{yz} + \sigma_{xz}),$$

$$\gamma = \frac{1}{E_z}(1 - \sigma_{yz}\,\sigma_{zy}). \qquad (7)$$

Within each layer of the sandwich the equations of equilibrium of the stress components X_x, X_z, and Z_z assure the existence of a stress function F such that

$$X_x = \frac{\partial^2 F}{\partial z^2}, \quad Z_z = \frac{\partial^2 F}{\partial x^2}, \quad X_z = -\frac{\partial^2 F}{\partial x \partial z} \qquad (8)$$

Substituting (8) in (2) and (6), and then making use of the compatibility equation

$$\frac{\partial^2 e_{xx}}{\partial z^2} + \frac{\partial^2 e_{zz}}{\partial x^2} = \frac{\partial^2 e_{xz}}{\partial x\,\partial z},$$

it follows that the stress function F satisfies the differential equation[6]

[5] U. S. Forest Products Laboratory Report No. 1312, p. 34.

[6] In printing equation (35) of the paper "Bending of a Centrally Loaded Rectangular Strip of Plywood," Physics 7, 1936, pages 32-41, the coefficients α and γ were interchanged. Opportunity is taken to point out an error that crept into the writing of equation (20) of that paper. The term N cos $\delta_n b$ in the denominator of the long fraction should be replaced by N sinh $\gamma_n b$. The calculations given were based on the correct expression.

$$\gamma \frac{\partial^4 F}{\partial x^4} + \left(\frac{1}{\mu_{xz}} - 2\beta\right) \frac{\partial^4 F}{\partial x^2 \, \partial z^2} + \alpha \frac{\partial^4 F}{\partial z^4} = 0 \tag{9}$$

A suitable solution is

$$F = g \left(x - \frac{a}{2}\right) \left(\frac{z^3}{3} + e\, z\right) \tag{10}$$

Expressions of the form (2), (6), and (10) hold for each layer separately. Equation (10) will have the following forms in the core and facings 1 and 2, respectively.

$$F_c = g_c \left(x - \frac{a}{2}\right) \left(\frac{z^3}{3} + e_c z\right)$$

$$F_1 = g_1 \left(x - \frac{a}{2}\right) \left(\frac{z^3}{3} + e_1 z\right) \tag{10,a}$$

$$F_2 = g_2 \left(x - \frac{a}{2}\right) \left(\frac{z^3}{3} + e_2 z\right)$$

The constants that appear are to be determined by the conditions that hold on the planes separating the facings and the core, from the condition

$$\int_{-(p + f_2)}^{q + f_1} X_z \, dz = \frac{P}{2b}. \tag{11}$$

and from the conditions that

$$X_z = 0, \quad z = -(p + f_2) \quad \text{and} \quad z = (q + f_1) \tag{12}$$

It follows from (8) and (10,a) that in the core

$$(X_z)_c = -\, g_c \left(z^2 + e_c\right) \tag{13}$$

$$(X_x)_c = 2g_c \left(x - \frac{a}{2}\right) z \tag{14}$$

$$(Z_z)_c = 0 \tag{15}$$

For the facings 1 and 2 the subscripts c are to be replaced by 1 and 2, respectively.

Equations (2) and (6), together with (13), (14), and (15), give the following expressions for the components of strain in the core:

$$(e_{xx})_c = \frac{\partial u_c}{\partial x} = 2\alpha_c \, g_c \left(x - \frac{a}{2}\right) z \tag{16}$$

$$(e_{zz})_c = \frac{\partial w_c}{\partial z} = -2\beta_c\, g_c\, (x - \frac{a}{2})\, z \tag{17}$$

$$(e_{xz})_c = \frac{\partial u_c}{\partial z} + \frac{\partial w_c}{\partial x} = \frac{1}{\mu_c} X_z = -\frac{g_c}{\mu_c}(z^2 + e_c) \tag{18}$$

where u_c and w_c denote components of displacement in the core and μ_c denotes the value of the modulus of rigidity μ_{xz} in the core.

Again the corresponding equations for the facings are found by replacing the subscript c by 1 and 2, respectively.

From (16) and (17)

$$u_c = \alpha_c\, g_c\, (x - \frac{a}{2})^2\, z + r_c\, (z) \tag{19}$$

$$w_c = -\beta_c\, g_c\, (x - \frac{a}{2})\, z^2 + s_c\, (x) \tag{20}$$

where $r_c(z)$ and $s_c(x)$ are arbitrary functions which are to be determined, apart from linear terms, by substitution of (19) and (20) in (18). On substituting in (19) and (20) the functions determined in this way the following expressions for the components of the displacement in the core are obtained.

$$u_c = \alpha_c\, g_c\, (x - \frac{a}{2})^2\, z - \frac{g_c}{\mu_c}(\frac{z^3}{3} + e_c z) + \frac{\beta_c\, g_c}{3} z^3 + k_c z + m_c \tag{21}$$

$$w_c = -\beta_c\, g_c\, (x - \frac{a}{2})\, z^2 - \frac{\alpha_c\, g_c}{3}(x - \frac{a}{2})^3 - k_c x + n_c \tag{22}$$

By writing the subscripts 1 and 2, respectively, in place of c, the corresponding expressions for the components of displacement in the facings are obtained.

The condition that the component of displacement u shall be continuous at the plane, $z = q$, requires that

$$\begin{aligned} &\alpha_c\, g_c\, (x - \frac{a}{2})^2 q - \frac{g_c}{\mu_c}(\frac{q^3}{3} + e_c q) + \frac{\beta_c\, g_c}{3} q^3 + k_c q + m_c \\ &\quad = \alpha_1\, g_1\, (x - \frac{a}{2})^2 q - \frac{g_1}{\mu_1}(\frac{q^3}{3} + e_1 q) + \frac{\beta_1\, g_1}{3} q^3 + k_1 q + m_1 \end{aligned} \tag{23}$$

This relation is an identity in x. Hence

$$\alpha_c\, g_c = \alpha_1\, g_1 \tag{24}$$

and

$$- \frac{g_c}{\mu_c} \left(\frac{q^3}{3} + e_c q\right) + \frac{\beta_c\, g_c}{3} q^3 + k_c q + m_c$$

$$= - \frac{g_1}{\mu_1} \left(\frac{q^3}{3} + e_1 q\right) + \frac{\beta_1\, g_1}{3} q^3 + k_1 q + m_1 \tag{25}$$

The continuity of the component w at the plane $z = q$ requires that

$$- \beta_c\, g_c \left(x - \frac{a}{2}\right) q^2 - \frac{\alpha_c\, g_c}{3} \left(x - \frac{a}{2}\right)^3 - k_c x + n_c$$

$$= - \beta_1\, g_1 \left(x - \frac{a}{2}\right) q^2 - \frac{\alpha_1\, g_1}{3} \left(x - \frac{a}{2}\right)^3 - k_1 x + n_1 \tag{26}$$

This identity in x yields the further relations

$$\beta_c\, g_c\, q^2 + k_c = \beta_1\, g_1\, q^2 + k_1 \tag{27}$$

$$\beta_c\, g_c\, q^2 \frac{a}{2} + n_c = \beta_1\, g_1\, q^2 \frac{a}{2} + n_1 \tag{28}$$

The following equations, corresponding to (24), (25), (27), and (28), are obtained from the conditions that the components u and w are continuous at the plane $x = - p$.

$$\alpha_c\, g_c = \alpha_2\, g_2 \tag{29}$$

$$\frac{g_c}{\mu_c} \left(\frac{p^3}{3} + e_c p\right) - \frac{\beta_c\, g_c}{3} p^3 - k_c p + m_c$$

$$= \frac{g_2}{\mu_2} \left(\frac{p^3}{3} + e_2 p\right) - \frac{\beta_2\, g_2}{3} p^3 - k_2 p + m_2 \tag{30}$$

$$\beta_c\, g_c\, p^2 + k_c = \beta_2\, g_2\, p^2 + k_2 \tag{31}$$

$$\beta_c\, g_c\, p^2 \frac{a}{2} + n_c = \beta_2\, g_2\, p^2 \frac{a}{2} + n_2 \tag{32}$$

By comparing equations (24) and (29) and noting that $\alpha_1 = \alpha_2$ since the facings are made of the same material, it is seen that

$$g_1 = g_2 \tag{33}$$

It will be convenient to introduce the notation

$$\rho = \frac{\alpha_1}{\alpha_c} = \frac{(E_x)_c}{(E_x)_1} \frac{(1 - \sigma_{xy}\ \sigma_{yx})_1}{(1 - \sigma_{xy}\ \sigma_{yx})_c} \tag{34}$$

Then in accordance with (24)

$$g_c = \frac{\alpha_1}{\alpha_c} g_1 = \rho\ g_1 \tag{35}$$

Further

$$\beta_1 = \beta_2 \tag{36}$$

since the facings are made of the same material.

Hence in all of the preceding equations g_2 will be replaced by g_1, β_2 by β_1, α_2 by α_1, and g_c by ρg_1.

The condition that the component of shearing stress X_z is continuous at the planes z = q and z = - p requires that

$$\rho g_1\ (q^2 + e_c) = g_1\ (q^2 + e_1) \tag{37}$$

$$\rho g_1\ (p^2 + e_c) = g_1\ (p^2 + e_2) \tag{38}$$

Further, it follows from (12) that

$$(q + f_1)^2 + e_1 = 0 \tag{39}$$

$$(p + f_2)^2 + e_2 = 0 \tag{40}$$

Hence

$$e_1 = - (q + f_1)^2 \tag{41}$$

$$e_2 = - (p + f_2)^2 \tag{42}$$

On substituting (41) and (42) in (37) and (38), respectively, it is found that

$$e_c = - q^2 - \frac{1}{\rho} (2q\ f_1 + f_1^2) \tag{43}$$

$$e_c = - p^2 - \frac{1}{\rho} (2p f_2 + f_2^2) \tag{44}$$

It is clear that q, the distance from the neutral plane $z = 0$ to the junction of the core and the facing f_1, must be chosen so that the two expressions for e_c are equal.

By equating these expressions and recalling that $p = c - q$, it is found that

$$q = \frac{f_2^2 - f_1^2 + 2c f_2 + \rho c^2}{2 (f_1 + f_2 + \rho c)} \tag{45}$$

To complete the determinations of the constants that appear in the expressions for u_c, w_c, u_1, w_1, u_2, and w_2, the following conditions are imposed at the fixed end $x = 0$ of the cantilever forming the right-hand half of the beam.

$$w_c = 0 \qquad x = 0 \qquad z = 0 \tag{46}$$

$$u_1 = 0 \qquad x = 0 \qquad z = q + f_1 \tag{47}$$

$$u_2 = 0 \qquad x = 0 \qquad z = - (p + f_2) \tag{48}$$

Similar conditions of fixity were found to lead to satisfactory conclusions in the case of a plywood strip.[4]

From conditions (47) and (48) and equation (21) written with subscripts 1 and 2, respectively, it follows after a slight reduction, using (41) and (42), that

$$\alpha_1 g_1 \frac{a^2}{4} (q + f_1) + \frac{2}{3} \frac{g_1}{\mu_1} (q + f_1)^3 + \frac{\beta_1 g_1}{3} (q + f_1)^3$$

$$+ k_1 (q + f_1) + m_1 = 0 \tag{49}$$

$$- \alpha_1 g_1 \frac{a^2}{4} (p + f_2) - \frac{2}{3} \frac{g_1}{\mu_1} (p + f_2)^3 - \frac{\beta_1 g_1}{3} (p + f_2)^3$$

$$- k_2 (p + f_2) + m_2 = 0 \tag{50}$$

From (46) and (22) it is found that

$$n_c = - \frac{\alpha_c \rho g_1 a^3}{24} \tag{51}$$

Substitute k_c in terms of k_1 from (27) in (25) and k_c in terms of k_2 from (31) in (30) and substract, obtaining

$$m_1 - m_2 = \frac{g_1}{\mu_1}\left(\frac{q^3}{3} + \frac{p^3}{3} + e_1 q + e_2 p\right) - \frac{\rho g_1}{\mu_c}\left(\frac{q^3}{3} + \frac{p^3}{3} + e_c q + e_c p\right)$$

$$+ \frac{2}{3}\beta_1\, g_1\,(q^3 + p^3) - \frac{2}{3}\rho\beta_c\, g_1\,(q^3 + p^3) \tag{52}$$

From (27) and (31)

$$k_2 = k_1 + \beta_1\, g_1\,(q^2 - p^2) - \rho\beta_c\, g_1\,(q^2 - p^2) \tag{53}$$

Subtract (50) from (49) after substituting (53) for k_2 in (50) and obtain after some reduction

$$m_1 - m_2 = -\alpha_1\, g_1\, \frac{a^2}{4}\, h - \frac{2}{3}\frac{g_1}{\mu_1}\left[(q + f_1)^3 + (p + f_2)^3\right]$$

$$- \frac{\beta_1\, g_1}{3}\left[(q + f_1)^3 + (p + f_2)^3\right]$$

$$- k_1 h - \beta_1\, g_1\,(p + f_2)\,(q^2 - p^2) + \rho\beta_c\, g_1\,(p + f_2)\,(q^2 - p^2) \tag{54}$$

where $h = q + f_1 + p + f_2$.

Equate expressions for $m_1 - m_2$ in (52) and (50) and solve for k_1 and obtain after considerable reduction

$$k_1 = -g_1\left\{\frac{\alpha_1 a^2}{4} + \frac{1}{\mu_1 h}\left(qf_1^2 + pf_2^2 + \frac{2}{3}f_1^3 + \frac{2}{3}f_2^3\right)\right.$$

$$+ \frac{\beta_1}{h}\left(q^2 h + qf_1^2 + pf_2^2 + \frac{f_1^3}{3} + \frac{f_2^3}{3}\right)$$

$$- \frac{\rho\beta_c}{h}\left[\frac{2}{3}q^3 + q^2 p - \frac{1}{3}p^3 + (q^2 - p^2)\, f_2\right]$$

$$\left. - \frac{\rho}{\mu_c h}\left(\frac{q^3}{3} + \frac{p^3}{3} + e_c c\right)\right\} \tag{55}$$

To obtain the deflection at the center of the beam the displacement w_1 at the end $x = \frac{a}{2}$, of the cantilever will be calculated. This will be measured with reference to a point on the plane of the neutral axis at the middle of the beam. Consequently, the deflection of points on the neutral plane at the center of the beam will be numerically equal to the quantity w_1 calculated at the end $x = \frac{a}{2}$ of the cantilever.

In accordance with (22)

$$(w_1)_{x = \frac{a}{2}} = - k_1 \frac{a}{2} + n_1 \tag{56}$$

From (51) and (28)

$$n_1 = g_1 \left[\rho\beta_c \frac{a}{2} q^2 - \beta_1 \frac{a}{2} q^2 - \frac{\alpha_1 a^3}{24} \right] \tag{57}$$

On substituting (55) and (57) in (56) the following expression is obtained after some reduction:

$$\begin{aligned}(w_1)_{x = \frac{a}{2}} = g_1 \Big\{ & \frac{\alpha_1 a^3}{12} + \frac{a}{2\mu_1 h} (qf_1^2 + pf_2^2 + \tfrac{2}{3} f_1^3 + \tfrac{2}{3} f_2^3) \\ & + \frac{\beta_1 a}{2h} (qf_1^2 + pf_2^2 + \tfrac{1}{3} f_1^3 + \tfrac{1}{3} f_2^3) \\ & + \frac{\rho\beta_c a}{2h} (\tfrac{q^3}{3} + \tfrac{p^3}{3} + q^2 f_1 + p^2 f_2) \\ & - \frac{\rho a}{2\mu_c h} (\tfrac{q^3}{3} + \tfrac{p^3}{3} + e_c c) \Big\}\end{aligned} \tag{58}$$

and this expression can be further reduced to the form:

$$\begin{aligned}(w_1)_{x = \frac{a}{2}} = \frac{g_1 \alpha_1 a^3}{12} \Big\{ 1 + \frac{2}{a^2 h} \Big[& \frac{1}{\alpha_1 \mu_1} (3qf_1^2 + 3pf_2^2 + 2f_1^3 + 2f_2^3) \\ & + \frac{\beta_1}{\alpha_1} (3qf_1^2 + 3pf_2^2 + f_1^3 + f_2^3) + \frac{\rho\beta_c}{\alpha_1} (q^3 + p^3 + 3q^2 f_1 + 3p^2 f_2) \\ & - \frac{\rho}{\alpha_1 \mu_c} (q^3 + p^3 + 3e_c c) \Big] \Big\}\end{aligned} \tag{59}$$

The coefficient g_1 can be calculated from the condition (11).

$$\int_{-(p + f_2)}^{q + f_1} X_z dz = - g_1 \int_{-(p + f_2)}^{-p} (z^2 + e_2)\, dz - \rho g_1 \int_{-p}^{q} (z^2 + e_c)\, dz - g_1 \int_{q}^{(q + f_1)} (z^2 + e_1)\, dz$$

After performing the integrations and making use of (41), (42), (43), and (44) the right-hand member of this equation reduces to:

$$\frac{2g_1}{3}\left[3q^2f_1 + 3qf_1^2 + 3p^2f_2 + 3pf_2^2 + f_1^3 + f_2^3 + \rho\ (q^3 + p^3)\right]$$

By (11) this expression is equal to $\frac{P}{2b}$. Hence

$$g_1 = \frac{3P}{4b\left[3q^2f_1 + 3qf_1^2 + 3p^2f_2 + 3pf_2^2 + f_1^3 + f_2^3 + \rho\ (q^3 + p^3)\right]} \tag{60}$$

The denominator is closely related to D, the stiffness of the beam as calculated without correcting for the effect of shear deformation. For,

$$D = b\int_{-(p + f_2)}^{-p} \frac{(E_x)_2\ z^2}{\lambda_2}\,dz + b\int_{-p}^{q} \frac{(E_x)_c\ z^2}{\lambda_c}\,dz + b\int_{q}^{q + f} \frac{(E_x)_1\ z^2}{\lambda_1}\,dz$$

where

$$\lambda_2 = \lambda_1 = (1 - \sigma_{xy}\ \sigma_{yx})_1, \quad \lambda_c = (1 - \sigma_{xy}\ \sigma_{yx})_c \tag{60,a}$$

$(E_x)_2 = (E_x)_1$. After noting that

$$\frac{(E_x)_2}{\lambda_2} = \frac{(E_x)_1}{\lambda_1} = \frac{1}{\alpha_1} \quad \text{and} \quad \frac{(E_x)_c}{\lambda_c} = \frac{1}{\alpha_c}$$

the expression for D is readily reduced to

$$D = \frac{b}{3\alpha_1}\left[3q^2f_1 + 3qf_1^2 + 3p^2f_2 + 3pf_2^2 + f_1^3 + f_2^3 + \rho\ (q^3 + p^3)\right] \tag{61}$$

where in accordance with (34)

$$\rho = \frac{\alpha_1}{\alpha_c}$$

It follows from (60) and (61) that

$$g_1 = \frac{P}{4\alpha_1 D} \tag{62}$$

By using (62), equation (59) can be written in the form:

$$(w_1)_{x = \frac{a}{2}} = \frac{Pa^3}{48D}\left[1 + \eta\,\frac{h^2}{a^2}\right] \tag{63}$$

where

$$\begin{aligned}\eta = \frac{2}{h^3}\Bigg[&\frac{1}{\alpha_1\mu_1}\,(3qf_1^2 + 3pf_2^2 + 2f_1^3 + 2f_2^3)\\ &+ \frac{\beta_1}{\alpha_1}\,(3qf_1^2 + 3pf_2^2 + f_1^3 + f_2^3)\\ &+ \frac{\rho\beta_c}{\alpha_1}\,(q^3 + p^3 + 3q^2f_1 + 3p^2f_2)\\ &- \frac{\rho}{\alpha_1\mu_c}\,(q^3 + p^3 + 3e_c c)\Bigg]\end{aligned} \tag{64}$$

In this expression q and e_c are to be calculated by formulas (45) and (43). Further, $p = c - q$.

As will be seen from the steps taken to calculate it, the stiffness D is the stiffness that would be determined in a load-deflection test of a centrally loaded beam if a correction for shear deformation were not necessary. Equation (63) shows that the effective stiffness of a centrally loaded strip of sandwich is equal to D divided by $1 + \eta\, h^2/a^2$. Consequently,

$$\text{Effective stiffness} = \frac{D}{1 + \eta\dfrac{h^2}{a^2}} \tag{65}$$

Formula (64) for the factor η that expresses the effect of shear deformation in the core, can be written in a form more convenient for calculation by introducing the following notation:

Let

$$f_1 = f - \delta \quad \text{and} \quad f_2 = f + \delta \tag{66}$$

so that

$$\frac{f_1 + f_2}{2} = f \quad \text{and} \quad \frac{f_2 - f_1}{2} = \delta \tag{67}$$

Also,

$$h = c + f_1 + f_2 = c + 2f \tag{68}$$

and

$$f = \frac{h - c}{2} \tag{69}$$

Let

$$q = \frac{c}{2} + \phi \tag{70}$$

and

$$p = \frac{c}{2} - \phi \tag{71}$$

Then it follows from equation (45), the relation $p = c - q$, and from equations (67) and (68) that

$$\phi = \frac{\delta h}{h - c + \rho c} \tag{72}$$

The following expressions that are needed in transforming (64) are obtained from equations (66) to (72).

$$f_2^2 - f_1^2 = 4\, f\, \delta$$

$$f_1^2 + f_2^2 = 2f^2 + 2\delta^2 = \frac{(h - c)^2}{2} + 2\delta^2$$

$$f_1^3 + f_2^3 = \frac{(h - c)^3}{4} + 3\,(h - c)\,\delta^2$$

$$qf_1^2 + pf_2^2 = \frac{c\,(h - c)^2}{4} + c\,\delta^2 - 2\,(h - c)\,\delta\phi$$

$$q^2 f_1 + p^2 f_2 = \frac{c^2\,(h - c)}{4} - 2c\,\delta\phi + (h - c)\,\phi^2$$

$$q^3 + p^3 = \frac{c^3}{4} + 3c\,\phi^2$$

After introducing these expressions in (64) for η and replacing ϕ by its value given in (72), after considerable reduction a formula is found for η that it is convenient to break up into four parts. Thus:

$$\eta = \eta_1 + \eta_2 + \eta_3 + \eta_4 \tag{73}$$

where

$$\eta_1 = \frac{1}{2\alpha_1\mu_1}\left\{2 - 3\frac{c}{h} + \frac{c^3}{h^3} - 12\,\frac{\delta^2}{h^2}\,\frac{c}{h}\,\frac{\left[1 - 2\rho - (1 - \rho)\frac{c}{h}\right]}{\left[1 - (1 - \rho)\frac{c}{h}\right]}\right\} \qquad (74)$$

$$\eta_2 = \frac{\beta_1}{2\alpha_1}\left\{1 - 3\,\frac{c^2}{h^2} + 2\frac{c^3}{h^3} - 12\,\frac{\delta^2}{h^2}\,\frac{\left[1 - (1 + \rho)\,\frac{c}{h}\right]}{\left[1 - (1 - \rho)\,\frac{c}{h}\right]}\right\} \qquad (75)$$

$$\eta_3 = \frac{\beta_c}{2\alpha_c}\left\{3\,\frac{c^2}{h^2} - 2\,\frac{c^3}{h^3} + 12\,\frac{\delta^2}{h^2}\,\frac{\left[1 - 2\frac{c}{h} + 2\,(1 - \rho)\,\frac{c^2}{h^2}\right]}{\left[1 - (1 - \rho)\,\frac{c}{h}\right]^2}\right\} \qquad (76)$$

$$\eta_4 = \frac{1}{2\alpha_1\mu_c}\left\{3\,\frac{c}{h}\,(1 - \frac{c^2}{h^2}) - 12\,\frac{\delta^2}{h^2}\,\frac{c}{h}\,\frac{\left[1 + (1 - \rho)\,\frac{c}{h}\right]}{\left[1 - (1 - \rho)\,\frac{c}{h}\right]} + 2\rho\frac{c^3}{h^3}\right\} \qquad (77)$$

The part η_4 will often be found to be a sufficiently good approximation to the value of η because the factor $\frac{1}{\alpha_1\mu_c}$ in the coefficient of the expression for η_4 is usually very large in comparison with the corresponding factors in the coefficients of the expressions for η_1, η_2, and η_3. Further, the last term in the brackets in the expression for η_4 will often be negligible. When this is the case, the approximate value of η is given by the equation:

$$\eta = \frac{1}{2\alpha_1\mu_c}\left\{3\,\frac{c}{h}\,(1 - \frac{c^2}{h^2}) - 12\,\frac{\delta^2}{h^2}\,\frac{c}{h}\,\frac{\left[1 + (1 - \rho)\,\frac{c}{h}\right]}{\left[1 - (1 - \rho)\,\frac{c}{h}\right]}\right\} \qquad (78)$$

It is difficult to state the limits within which the formula (78), or the somewhat more accurate formula (77), is a sufficiently good approximation. With a given type of sandwich it will be necessary to consider the other terms in formula (73) in order to decide the limits within which either formula (78) or (77) may be used to calculate η. The computation for a series of similar specimens will be materially shortened if it can be shown that either of the simpler formulas is adequate for the series.

In the notation introduced in equations (66) and (67) the formula (61) for the stiffness D, not corrected for shear deformation, takes the following simpler form:

$$D = \frac{b}{12\alpha_1}\left[h^3 - c^3 + \rho c^3 - \frac{12c\ \delta^2\ (1 - \rho)}{1 - (1 - \rho)\frac{c}{h}}\right] \tag{79}$$

where α_1, ρ, and δ are defined by equations (7), (31), and (67), respectively.

Facings of Equal Thickness

When the facings are of equal thickness, the formulas (74) - (77) for the components η_1, η_2, η_3, and η_4 take much simpler forms. For if $f_1 = f_2$, it follows from (67) that $\delta = 0$. Consequently, all terms containing δ as a factor disappear from equations (74) - (77). At this point the necessary formulas are collected for the case of equal facings.

The central deflection is given by:

$$(w_1)_{x = \frac{a}{2}} = \frac{Pa^3}{48D}\left[1 + \eta\frac{h^2}{a^2}\right] \tag{63}$$

where

$$\eta = \eta_1 + \eta_2 + \eta_3 + \eta_4 \tag{73}$$

and

$$\eta_1 = \frac{1}{2\alpha_1\mu_1}\left[2 - 3\frac{c}{h} + \frac{c^3}{h^3}\right] \tag{80}$$

$$\eta_2 = \frac{\beta_1}{2\alpha_1}\left[1 - 3\frac{c^2}{h^2} + 2\frac{c^3}{h^3}\right] \tag{81}$$

$$\eta_3 = \frac{\beta_c}{2\alpha_c}\left[3\frac{c^2}{h^2} - 2\frac{c^3}{h^3}\right] \tag{82}$$

$$\eta_4 = \frac{1}{2\alpha_1\mu_c}\left[3\frac{c}{h}\left(1 - \frac{c^2}{h^2}\right) + 2\rho\frac{c^3}{h^3}\right] \tag{83}$$

In the original form of this report the thickness of the core was denoted by t. The change from t to c has been made here for the sake of agreement with the notation of more recent reports.

Formulas (63) and (73) combined with equations (80), (81), (82), and (83) are substantially the same as formulas (53) and (54) of the original form of this report, except that in the present report the factor $\lambda = 1 - \sigma_{xy}\,\sigma_{yx}$, both for facings and core, has not been replaced by unity but has been retained throughout.

For sandwich beams with facings of equal thickness, such as were considered in the original report, Hoff and Mautner[1] have derived by an entirely different method an approximate formula expressing the effect of the shear deformation in the core on the central deflection of a centrally loaded beam. Numerical calculations of a few beams indicate reasonable agreement of their formula, applicable to the case of equal facings, with formula (73) combined with equations (80), (81), (82), and (83) and with the simpler approximate formula to be mentioned below.

As in the case of unequal facings, it will often be found that η_4 is a sufficiently good approximation to η. Often a further approximation can be made by dropping the term $2\rho c^3/h^3$. When this is permissible, the following simple approximate formula for η results:

$$\eta = \frac{3}{2\alpha_1\mu_c}\,\frac{c}{h}\left(1 - \frac{c^2}{h^2}\right) \tag{84}$$

or, in terms of the elastic constants, this approximate formula can be written as:

$$\eta = \frac{3\,E_{x1}}{2(1 - \sigma_{xy}\,\sigma_{yx})_1\,\mu_{zxc}}\,\frac{c}{h}\left(1 - \frac{c^2}{h^2}\right) \tag{85}$$

where the subscript 1 denotes an elastic constant of the facings and the subscript c an elastic constant of the core.

As in the case of the approximate formulas for sandwiches with unequal facings, it will be necessary to refer to the more exact formula to determine the limits within which η_1, η_2, and η_3 may be neglected for a given type of sandwich.

For sandwiches with facings of equal thickness, the stiffness D from which the effective stiffness can be calculated by equation (65) can be obtained by using equation (79) and setting δ equal to zero.

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[1] Hoff, N. J., and Mautner, S. E. Bending and Buckling of Sandwich Beams. Jour. Aeronaut. Sciences 15, 707, 1948.

Notation

a	span
b	width of strip
c	thickness of core
D	flexural rigidity of strip, neglecting the effect of shear deformation in the core
e_1, e_2, e_c	constants in expressions for the stress functions Equation (10,a)
e_{xx}, e_{yy}, ...	components of strain
E_x, E_y, E_z	Young's moduli
f_1, f_2	thickness of the facings
f	$f = (f_1 + f_2)/2$, average thickness of the facings
g_1, g_2, g_c	constants in expressions for the stress functions Equation (10,a)
h	thickness of strip
k, m, n	constants of integration
p	distance from the neutral axis to the facing f_2
P	load applied at mid-span
q	distance from the neutral axis to the facing f_1
X_x, Y_y, ...	components of stress
u, v, w	components of displacement
α, β, γ	combinations of constants of facings or core Equations (7)
δ	$\delta = (f_2 - f_1)/2$
η	factor showing influence of shear deformation. Equations (63), (64), (73), (77), (78), (83), and (84).
λ	combination involving Poisson's ratios. Equation (60,a)
μ_{xz}	modulus of rigidity
μ_c	μ_{xz} in core
μ_1	μ_{xz} in facings
ρ	ratio of constants of facings and core. Equation (34)
σ_{xy} σ_{yz} ...	Poisson's ratios

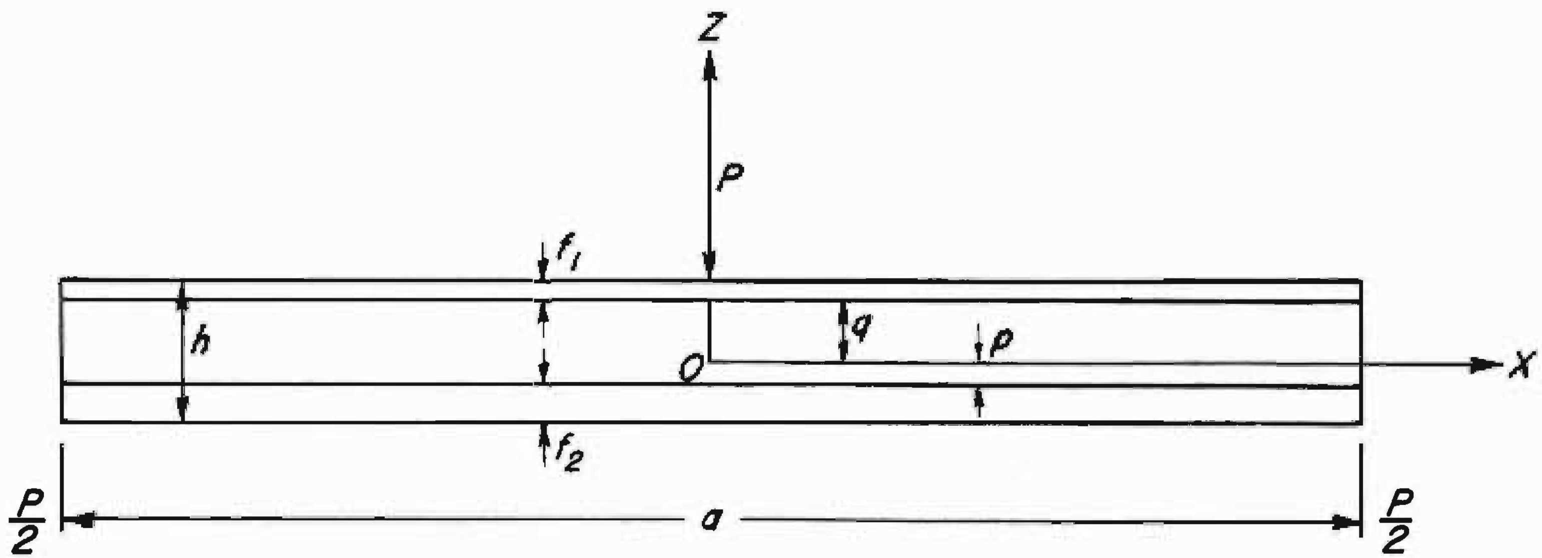

Figure 1.--Rectangular strip, centrally loaded.

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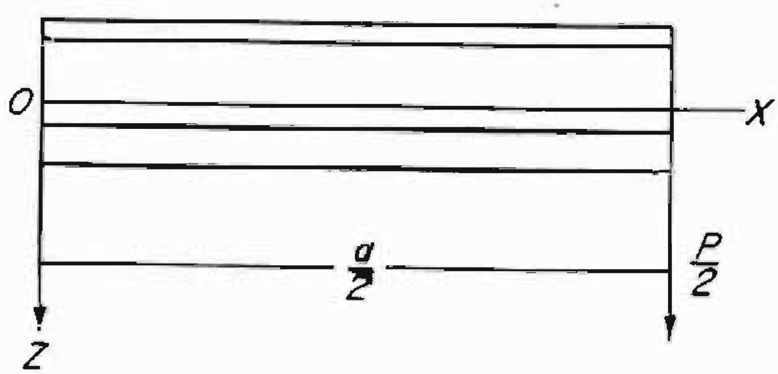

Figure 2.--Half of strip as a cantilever.

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