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On the New Evaluation of an Old Integral

by William Walters and Michael Huber

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On the New Evaluation of an Old Integral

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14. ABSTRACT
A simple technique is shown to evaluate $\int \frac{dx}{x^m (1+x^n)}$.

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1. Introduction

Consider the integral

$$\int \frac{dx}{x^m (1+x)}. \tag{1}$$

In the *CRC Standard Mathematical Tables,*[1] equation 1 can require repeated integral evaluations. Integral 87 shows

$$\int \frac{dx}{x^m (a+bx^n)^{p+1}} = \frac{1}{a}\int \frac{dx}{x^m (a+bx^n)^p} - \frac{b}{a}\int \frac{dx}{x^{m-n} (a+bx^n)^{p+1}}. \tag{2}$$

Enter this integral into your favorite computer algebra system. *Maple* returns

$$\int \frac{dx}{x^m (1+x)} = \frac{x^{-m}(m-1)}{(1-m)m} + \frac{x^{-m}(m-1)\,\mathrm{LerchPhi}(-x,1,-m)}{1-m}, \tag{3}$$

which involves Lerch's Phi transcendent function, defined as

$$\mathrm{LerchPhi}(x,a,m) = \sum_{\mathrm{n}=0}^{\infty} \frac{\mathrm{x}^{\mathrm{n}}}{(\mathrm{m}+\mathrm{n})^{\mathrm{a}}}. \tag{4}$$

Similarly, *Mathematica* gives

$$\int \frac{dx}{x^m (1+x)} = -\frac{x^{1-m}\ \mathrm{Hypergeometric\ 2F1}[1-m, 1, 2-m, x]}{-1+m}, \tag{5}$$

which involves knowing the implementation of Hypergeometric functions. In this report, we seek to provide a simpler evaluation for integrals of this form (equation 1). We state up front that the exponent m need not be an integer.

2. Using Infinite Sums

The integrand of equation 1 can be rewritten as follows:

$$\frac{x^{-m}}{1+x}. \tag{6}$$

By long division,

[1] Beyer, W. H., Ed. *CRC Standard Mathematical Tables*. CRC Press, Inc.: West Palm Beach, FL, 1978.

$$\frac{x^{-m}}{1+x} = x^{-m} - x^{-m+1} + x^{-m+2} - x^{-m+3} + \cdots, \tag{7}$$

which is an infinite sum in the form

$$\sum_{i=0}^{\infty} (-1)^i \, x^{-m+i}. \tag{8}$$

Therefore, the integral becomes

$$\begin{aligned} \int \frac{dx}{x^m (1+x)} &= \int \sum_{i=0}^{\infty} (-1)^i \, x^{-m+i} dx \\ &= \sum_{i=0}^{\infty} \int (-1)^i \, x^{-m+i} dx \\ &= \sum_{i=0}^{\infty} \frac{(-1)^i \, x^{-m+i+1}}{-m+i+1} + C, \end{aligned} \tag{9}$$

for some constant C. Noting that

$$\frac{1}{1+x} = \sum_{i=0}^{\infty} (-1)^i \, x^i \tag{10}$$

converges for $x^2 < 1$ allows us to rewrite equation 9 as

$$\int \frac{dx}{x^m (1+x)} = \frac{1}{x^m} \sum_{i=0}^{\infty} \frac{(-1)^i \, x^{i+1}}{i-m+1} + C; \tag{11}$$

again, m need not be an integer. However, if m = 1, 2, 3, … , then for $i = m–1$, $x^{-m+i} = x^{-1}$, so

$$\int \frac{dx}{x} = \ln x \tag{12}$$

for the $i = m - 1$ term, or

$$\int \frac{dx}{x^m (1+x)} = \frac{1}{x^m} \sum_{i=0}^{m-2} \frac{(-1)^i \, x^{i+1}}{i-m+1} + \ln x + \sum_{i=m}^{\infty} \frac{(-1)^i \, x^{i+1}}{i-m+1} + C. \tag{13}$$

This allows us to sum past the singularity.

3. A Generalization

Generalizing, consider the following:

$$\int \frac{dx}{x^m\,(1+x^n)} = \int \frac{x^{-m}\,dx}{1+x^n}. \tag{14}$$

From long division, the integrand can be rewritten as $x^{-m}\,(1 - x^n + x^{2n} - x^{3n} + \ldots)$,

or

$$\frac{x^{-m}}{1+x^n} = \sum_{i=0}^{\infty} (-1)^i\; x^{in-m}. \tag{15}$$

Thus,

$$\int \frac{dx}{x^m\left(1+x^n\right)} = \int \sum_{i=0}^{\infty} \left(-1\right)^i x^{in-m} dx = \sum_{i=0}^{\infty} \frac{\left(-1\right)^i\; x^{in-m+1}}{ni-m+1} + C\,, \tag{16}$$

and we sum prior to integration past any singularity. As before, m and n need not be integers, and this technique converges for | x | < 1.

4. Examples

As a first example, we consider the case with $m = 3$, an integer. We start with equation 2 (shown previously), where $m = 3$, $a = b = n = 1$, and $p = 0$. Continuing, we find that

$$\int \frac{dx}{x^3(1+x)} = \int \frac{dx}{x^3} - \int \frac{dx}{x^2(1+x)}$$

$$= -\frac{x^{-2}}{2} - \left[\int \frac{dx}{x^2} - \int \frac{dx}{x(1+x)} \right]$$

$$= -\frac{x^{-2}}{2} + x^{-1} + \ln x - \ln\,(1+x) + C$$

$$= -\frac{1}{2x^2} + \frac{1}{x} + \ln x - \left[x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \right] + C$$

$$= -\frac{1}{2x^2} + \frac{1}{x} + \ln x - x + \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4} - \cdots + C\;, \tag{17}$$

which is valid for $-1 < x < 1$. Using our approach and equation 9,

$$\begin{aligned}\int \frac{dx}{x^3(1+x)} &= \int \sum_{i=0}^{\infty} (-1)^i \, x^{-3+i} dx \\ &= \int \left[x^{-3} - x^{-2} + x^{-1} - x^0 + x^1 - \cdots\right] dx \\ &= -\frac{1}{2x^2} + \frac{1}{x} + \ln x - x + \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4} - \cdots + C,\end{aligned} \tag{18}$$

which matches the CRC's solution. When m is a positive integer, equation 2 is adequate. However, as m gets large, the CRC's proposal will involve evaluating several integrals. Our technique is much simpler.

Now, suppose $m = \frac{1}{5}$, $n = 3$, and we keep $a = b = 1$ and $p = 0$. Equation 2 becomes

$$\int \frac{dx}{x^{\frac{1}{5}}(1+x^3)} = \int \frac{dx}{x^{\frac{1}{5}}} - \int \frac{dx}{x^{\frac{-14}{5}}(1+x^3)} . \tag{19}$$

While the first integral on the right-hand side is easy enough to evaluate, the second is not trivial, unless the process is repeated or the integrand is expanded in a power series. Turning to equation 15, we find

$$\begin{aligned}\int \frac{dx}{x^{\frac{1}{5}}(1+x^3)} &= \int \sum_{i=0}^{\infty} (-1)^i \, x^{3i-\frac{1}{5}} dx \\ &= \int \left[x^{-\frac{1}{5}} - x^{\frac{14}{5}} + x^{\frac{29}{5}} - x^{\frac{44}{5}} + \cdots \right] dx \\ &= \frac{5}{4} x^{\frac{4}{5}} - \frac{5}{19} x^{\frac{19}{5}} + \frac{5}{34} x^{\frac{34}{5}} - \frac{5}{49} x^{\frac{49}{5}} + \cdots + C.\end{aligned} \tag{20}$$

We do not offer any practical applications for the integral. However, the proposed method converges for $x^2 < 1$ or when the independent variable is normalized, to guarantee that the normalized value of x is less than 1 when nondimensionalized; otherwise, the series diverges.

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