ARMY RESEARCH LABORATORY

Interval Estimates for Probabilities of Non-Perforation Derived From a Generalized Pivotal Quantity

by David W. Webb

ARL-TR-3596 **September 2005**

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A generalized pivotal quantity is developed that yields exact confidence intervals for the cumulative distribution function (CDF) at a specific value when the underlying distribution is assumed to be normal. This problem is similar to the development of a tolerance interval, and unsurprisingly, its solution involves the non-central "t" distribution. Confidence bands for a normal CDF follow easily. Military applications include vulnerability and lethality assessment (e.g., interval estimation for the probability of non-perforation against homogeneous armored targets).

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1. Introduction

The estimation of non-perforation probabilities is of significant interest to the developers of armor systems, whose objective is to provide protection for vehicles and personnel against enemy threats. In particular, the following questions may arise:

1. What is the probability that homogenous armor of a given thickness will not be perforated by a specific enemy threat?

2. How thick should the armor be if one is to be highly confident that the probability of non-perforation with this threat is high?

Conversely, these probabilities are of interest to the developers of armor-piercing projectiles, whose objective is to perforate enemy armor. From this point of view, one might ask

3. What is the probability that a projectile will be able to perforate an enemy armor of a given thickness?

For a particular projectile, non-perforation[1] occurs if its penetration depth, denoted by *x,* is less than the thickness of the armor, denoted by x_0. Now consider the random variable *X*, which represents the penetration depth of a randomly selected projectile. The probability of non-perforation is an unknown, fixed constant within the interval [0, 1] and is written as $P(X \le x_0)$. An estimate for the true non-perforation probability will be based on sample data and as such, is subject to sampling variation. However, a point estimate for the non-perforation probability is a single value and therefore does not give any sense of the uncertainty associated with it. Preferable to a point estimate is an interval estimate that contains the non-perforation probability with high probability, i.e., a confidence interval. The width of the interval provides a quantitative measure of the estimation error.

The problem of estimating $P(X \le x_0)$ has its origins in acceptance sampling plans for statistical quality control in the 1950's. Lieberman and Resnikoff (1955) and Barton (1961) yielded expressions for the minimum variance estimator. It was not until the 1970's that confidence interval estimators surfaced. In their 1977 paper, Owen and Hua derived lower confidence limits for $P(X \ge x_0)$ and described how to obtain one-sided and two-sided limits for $P(X \le x_0)$. However, their derivation contains several errors in notation and as a result, is quite difficult to follow. Owen and Hua's paper also provided tables that assist in the calculation of lower confidence limits for $P(X \ge x_0)$; these tables were dramatically expanded in Odeh and Owen

[1] For the remainder of this report, we only discuss non-perforation events, their probabilities of occurrence, and interval estimates for these probabilities. Analogous results for the probability of perforation, $P(X > x_0)$, are easily derived if we use the axiomatic property $P(X > x_0) = 1 - P(X \le x_0)$.

(1980, tables 7.1 through 7.7). Odeh and Owen's tables are cited in the more recent engineering statistics text by Hahn and Meeker (1991, section 4.5).

In this report, the recent concept of generalized inference is used to develop one- and two-sided confidence intervals for the probability of non-perforation when it can be assumed that the penetrations follow a normal distribution. Although this may be seen as an attempt to "reinvent the wheel," it is meant to show the wide applicability of the generalized approach to confidence interval construction. As pointed out by Weerahandi (2004, section 1.7), generalized inference may be used to obtain confidence intervals for *any* function of the normal distribution parameters.

Solutions for the limits of the intervals require iterative calculation of percentiles of non-central t distributions. As an alternative to the tables by Odeh and Owen and to avoid the interpolation errors inherent in their usage, software code is provided in the appendices so that the practitioner can easily obtain exact solutions for the interval limits.

2. General Solution

Suppose that *n* depths of penetration $x_1, x_2, \ldots, x_n$ are collected in an experiment for which the observations are assumed to come from a normal distribution with mean μ and standard deviation σ. Historically, this assumption has been found to be tenable when the projectiles are of the same ammunition type with (approximately) equal muzzle velocities, obliquities, etc., and the target is a homogeneous armor. For the purpose of this report, we assume that the armor is thick enough to guarantee that no projectiles will perforate. From this sample, an estimate is desired for the probability of non-perforation, i.e., the probability that a future penetration, *X*, from the same population of projectiles will be less than or equal to some specified depth, x_0. This probability is written as $P(X \le x_0)$ and is known as the cumulative distribution function (CDF) evaluated at x_0, or $F_X(x_0)$. Notice that

$$P(X \le x_0) = P\left(\frac{X-\mu}{\sigma} \le \frac{x_0-\mu}{\sigma}\right) = P\left(Z \le \frac{x_0-\mu}{\sigma}\right) = \Phi\left(\frac{x_0-\mu}{\sigma}\right), \tag{1}$$

in which *Z* is a standard normal random variable and $\Phi(\cdot)$ is the standard normal cumulative distribution function (CDF) commonly tabulated in statistics texts and available in many software packages. Therefore, calculating a simple point estimate for $P(X \le x_0)$ is straightforward: one computes the standard normal CDF at an estimate of $\frac{x_0-\mu}{\sigma}$. We obtain this estimate by substituting the sample mean ($\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$) and the sample standard deviation

($s = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2}$) for the population mean (μ) and the population standard deviation (σ), respectively[2]. That is,

$$\hat{F}_X(x_0) = \hat{P}(X \le x_0) = \hat{P}\left(Z \le \frac{x_0 - \mu}{\sigma}\right) \approx P\left(Z \le \frac{x_0 - \bar{x}}{s}\right) = \Phi\left(\frac{x_0 - \bar{x}}{s}\right). \tag{2}$$

However, any simple point estimate for $F_X(x_0)$ does not give information about its precision. What we would prefer to construct is a confidence interval or perhaps a confidence bound for $F_X(x_0)$ that by its width gives some sense of the degree of uncertainty associated with it.[3]

We begin by recalling from equation 1 that $F_X(x_0) = \Phi\left(\frac{x_0 - \mu}{\sigma}\right)$. Since $\Phi(\cdot)$ is a strictly monotonic function, we can easily attain a confidence interval for $\Phi\left(\frac{x_0 - \mu}{\sigma}\right)$ by deriving a confidence interval for $\frac{x_0 - \mu}{\sigma}$ and then applying the function $\Phi(\cdot)$ to the resulting confidence limits (see Mood, Graybill, and Boes, 1974, page 378).

The following derivation of a confidence interval for $\theta = \frac{x_0 - \mu}{\sigma}$ is based on a method by Weerahandi (1995) in which he uses a generalized pivotal quantity to obtain a confidence interval for the coefficient of variation, $\frac{\sigma}{\mu}$, for a normal population with mean μ and standard deviation σ. The construction of a generalized pivotal quantity requires one to find a function *R* with arguments

1. $\bar{X}$ and S^2 (independent and sufficient statistics for the random sample $X_1, \ldots, X_n$),
2. $\bar{x}$ and s^2 (the observed values of $\bar{X}$ and S^2),
3. θ, the parameter of interest, and
4. perhaps a vector, ξ, of additional unknown (nuisance) parameters.

---

[2]Although this estimate is easily obtained, it does not have the desirable property of being unbiased. An unbiased estimator with minimum variance is presented in either Lieberman and Resnikoff (1955) or Barton (1961).

[3]Note the distinction between the problems of solving for a confidence interval for $\Phi\left(\frac{x_0 - \mu}{\sigma}\right)$ and solving for a tolerance interval. With the former, the percentile value (x_0) is specified and an interval for its associated CDF value, $\Phi\left(\frac{x_0 - \mu}{\sigma}\right)$, is desired. On the other hand, with tolerance interval construction, the CDF value (*p*) is specified while an interval for the associated percentile, $\mu + z_p\sigma$, is desired.

This function, denoted in full by $R(\overline{X}, S^2; \overline{x}, s^2, \theta, \xi)$, must satisfy the following two conditions:

Condition 1: $R(\overline{X}, S^2; \overline{x}, s^2, \theta, \xi)$ has a distribution that is free of any unknown parameters.

Condition 2: The observed value of *R*, i.e., $r = R(\overline{x}, s^2; \overline{x}, s^2, \theta, \xi)$, is equal to θ.

If such a function can be found, then it is a generalized pivotal quantity for θ, and its percentiles can be used to obtain confidence intervals for θ. Since their introduction in the 1980's, generalized pivotal quantities have been regarded as quite challenging to derive. Even Weerahandi (1993) states "… the construction of pivotals requires some intuition." In essence, one obtains the pivotal quantity by working backwards from the expression for θ. The mathematical "tricks" of adding 0 and/or multiplying by 1 are adroitly employed to link all unknown parameters in θ with functions of the sufficient statistics whose distributions are parameter free. Once the linking is complete, all remaining random variables are converted to their observed values.

<u>Step 1</u>: The sufficient statistics for a normally distributed random sample are $\overline{X}$ and S^2. Random variables based on these statistics whose distributions are free of unknown parameters include

1. $Y = \dfrac{\overline{X} - \mu}{\sigma}$, distributed as a normal random variable with mean zero and variance $\dfrac{1}{n}$; and
2. $V = (n-1)\dfrac{S^2}{\sigma^2}$, distributed as a chi-square random variable with $n-1$ degrees of freedom.

<u>Step 2:</u> We attempt to construct a random variable from θ that involves *Y* and/or *V*. First, notice that by adding zero in the form of $(-\overline{X} + \overline{X})$ to the numerator, θ can be rewritten in the following manner:

$$\theta = \frac{x_0 - \mu}{\sigma} = \frac{x_0 - \overline{X} + \overline{X} - \mu}{\sigma} = \frac{x_0 - \overline{X}}{\sigma} + \frac{\overline{X} - \mu}{\sigma}. \tag{3}$$

Now the parameter μ is linked to the random variable $\dfrac{\overline{X} - \mu}{\sigma}$ which was previously noted as having a parameter-free distribution. When we make the substitution $Y = \dfrac{\overline{X} - \mu}{\sigma}$, the parameter μ is removed from θ:

$$\theta = \frac{x_0 - \overline{X}}{\sigma} + Y. \tag{4}$$

It remains to manipulate the left addend of equation 4 so that the unknown parameter σ is linked to a random variable whose distribution is free of unknown parameters. We can achieve this by multiplying the left addend by 1, in the form of $\sqrt{\frac{(n-1)S^2}{(n-1)S^2}}$:

$$\theta = \frac{x_0 - \overline{X}}{\sigma} + Y = \frac{x_0 - \overline{X}}{\sigma}\sqrt{\frac{(n-1)S^2}{(n-1)S^2}} + Y = \frac{x_0 - \overline{X}}{S\sqrt{n-1}}\sqrt{\frac{(n-1)S^2}{\sigma^2}} + Y. \tag{5}$$

Now σ is linked to $\frac{(n-1)S^2}{\sigma^2}$ which was noted in step 1 as having a parameter-free distribution. After $V = (n-1)\frac{S^2}{\sigma^2}$ is substituted in equation 5, all unknown parameters are removed from θ:

$$\theta = \frac{x_0 - \overline{X}}{S\sqrt{n-1}}\sqrt{\frac{(n-1)S^2}{\sigma^2}} + Y = \frac{x_0 - \overline{X}}{S\sqrt{n-1}}\sqrt{V} + Y. \tag{6}$$

<u>Step 3</u>: At this point, we replace the remaining sufficient statistics in equation 6 with their observed values; that is, $\frac{x_0 - \overline{x}}{s\sqrt{n-1}}\sqrt{V} + Y$.

<u>Step 4</u>: Finally, we rewrite the random variables V and Y in their original form (as functions of the sufficient statistics) to obtain the generalized pivotal quantity:

$$R\left(\overline{X}, S^2; \overline{x}, s^2, \mu, \sigma\right) = \frac{x_0 - \overline{x}}{s\sqrt{n-1}}\sqrt{\frac{(n-1)S^2}{\sigma^2}} + \frac{\overline{X} - \mu}{\sigma}. \tag{7}$$

As was demonstrated in the development of R, its distribution is free of any unknown parameters. Thus, condition 1 is satisfied. Furthermore, the observed value of R is

$$r = R\left(\overline{x}, s^2; \overline{x}, s^2, \mu, \sigma\right) = \frac{x_0 - \overline{x}}{s}\sqrt{\frac{s^2}{\sigma^2}} + \frac{\overline{x} - \mu}{\sigma} = \theta,$$

thus satisfying condition 2. Therefore, R is a generalized pivotal quantity whose percentiles can be used to obtain a confidence interval for θ. Using the normal cumulative distribution, $\Phi(\cdot)$, we then ultimately obtain a desired confidence interval for the probability of non-perforation.

3. Lower Confidence Bound (LCB)

Since effective armor offers a high probability of non-perforation, engineers and management will most often be interested in a lower confidence bound, since this would represent a "pessimistic" bound on the true probability of non-perforation, $\Phi\left(\frac{x_0-\mu}{\sigma}\right)$.

A $(1-\alpha)100\%$ LCB for $\theta=\frac{x_0-\mu}{\sigma}$ is that value B_L for which

$$1-\alpha = P(B_L \le R) \tag{8}$$

is satisfied.[4] We can estimate a generalized LCB for θ by randomly generating a large number of observations of R and selecting a value for which equation 5 is empirically satisfied. For instance, with $\alpha = .05$, one could start by generating 100,000 observations of R, denoted by $r_1, r_2, \ldots, r_{100000}$. If the ordered observations are written as $r_{(1)} < r_{(2)} < \ldots < r_{(100000)}$, then a 95% LCB for θ would be any real value B_L so that $r_{(5000)} < B_L < r_{(5001)}$, such as $B_L = \frac{r_{(5000)} + r_{(5001)}}{2}$.

However, equation 8 can be expanded to get

$$1-\alpha = P\left(B_L \le \frac{x_0-\bar{x}}{s}\sqrt{\frac{S^2}{\sigma^2}} + \frac{\bar{X}-\mu}{\sigma}\right) = P\left(B_L \le \frac{x_0-\bar{x}}{s}\sqrt{\frac{V}{n-1}} + Y\right). \tag{9}$$

Notice that the random variable Y in equation 9 is equal in probability to $Z/\sqrt{n}$, in which Z is a standard normal random variable (having mean 0 and variance 1). By using this fact and rearranging the terms in the last probability statement, we have

$$\begin{aligned}
1-\alpha &= P\left(\frac{x_0-\bar{x}}{s}\sqrt{\frac{V}{n-1}} + \frac{Z}{\sqrt{n}} \ge B_L\right) \\
&= P\left(\frac{x_0-\bar{x}}{s}\sqrt{\frac{n}{n-1}}\sqrt{V} + Z \ge \sqrt{n}B_L\right) \\
&= P\left(\frac{Z-\sqrt{n}B_L}{\sqrt{V/n-1}} \ge \frac{\bar{x}-x_0}{s}\sqrt{n}\right) \\
&= P\left(T_{n-1,-\sqrt{n}B_L} \ge \frac{\bar{x}-x_0}{s/\sqrt{n}}\right).
\end{aligned} \tag{10}$$

[4]It is implicit that the upper bound equals 1 in the probability statement of equation 8, i.e., $1 - \alpha = P(B_L \le R \le 1)$.

In equation 10, $T_{n-1,-\sqrt{n}B_L}$ is a non-central t random variable with $n-1$ degrees of freedom and non-centrality parameter $-\sqrt{n}B_L$ (see for example, Casella and Berger, 1990). However, a non-central t random variable with non-centrality parameter $-\sqrt{n}B_L$ is the mirror image of a non-central t random variable with non-centrality parameter $\sqrt{n}B_L$ (Johnson and Kotz, 1970). Therefore,

$$1-\alpha = P\left(T_{n-1,-\sqrt{n}B_L} \geq \frac{\bar{x}-x_0}{s/\sqrt{n}}\right) = P\left(T_{n-1,\sqrt{n}B_L} \leq \frac{x_0-\bar{x}}{s/\sqrt{n}}\right). \tag{11}$$

The final probability expression of equation 11 is the CDF of a non-central t random variable with $n-1$ degrees of freedom and non-centrality parameter $\sqrt{n}B_L$, i.e.,

$$1-\alpha = G_{n-1,\sqrt{n}B_L}\left(\frac{x_0-\bar{x}}{s/\sqrt{n}}\right). \tag{12}$$

A lower confidence bound for $\theta = \frac{x_0-\mu}{\sigma}$, is B_L, the solution to equation 12. When we exploit the strict monotonicity of the normal distribution function, a $(1-\alpha)100\%$ LCB for the probability of non-perforation, $\Phi\left(\frac{x_0-\mu}{\sigma}\right)$, is $\Phi(B_L)$.

4. Upper Confidence Bound (UCB)

A UCB would represent an "optimistic" bound on the true probability of non-perforation. As such, it is of little practical value and rarely calculated. However, its derivation is briefly discussed here for completeness and as a precursor to the development of a two-sided confidence interval.

Following a similar progression to that of the previous section, a $(1-\alpha)100\%$ UCB for $\theta = \frac{x_0-\mu}{\sigma}$ is that value B_U satisfying

$$1-\alpha = P(R \leq B_U) = P\left(T_{n-1,\sqrt{n}B_U} \geq \frac{x_0-\bar{x}}{s/\sqrt{n}}\right).$$

Therefore,

$$\alpha = G_{n-1,\sqrt{n}B_U}\left(\frac{x_0-\bar{x}}{s/\sqrt{n}}\right), \tag{13}$$

and a $(1-\alpha)100\%$ UCB for the probability of non-perforation, $\Phi\left(\frac{x_0-\mu}{\sigma}\right)$, is $\Phi(B_U)$.

5. Two-Sided Confidence Interval

A two-sided confidence interval might be of interest to the researcher simultaneously wanting lower and upper limits for the probability of non-perforation, neither of which is automatically set to their extreme value of 0 or 1, respectively.

A $(1-\alpha)100\%$ two-sided confidence interval for $\theta = \frac{x_0-\mu}{\sigma}$ is given by values C_L and C_U satisfying

$$\frac{\alpha}{2} = P(R \le C_L) \text{ and } \frac{\alpha}{2} = P(R \ge C_U)$$

Following steps similar to those used in determining one-sided confidence bounds, two-sided confidence limits for $\theta = \frac{x_0-\mu}{\sigma}$ are solutions to

$$1-\frac{\alpha}{2} = G_{n-1,\sqrt{n}C_L}\left(\frac{x_0-\bar{x}}{s/\sqrt{n}}\right), \tag{14}$$

and

$$\frac{\alpha}{2} = G_{n-1,\sqrt{n}C_U}\left(\frac{x_0-\bar{x}}{s/\sqrt{n}}\right). \tag{15}$$

Thus, a $(1-\alpha)100\%$ two-sided confidence interval for the probability of non-perforation, $\Phi\left(\frac{x_0-\mu}{\sigma}\right)$, is $(\Phi(C_L), \Phi(C_U))$.

6. Iterative Solutions for the Confidence Bounds and Confidence Limits

As an example, consider solving equation 12 for the LCB of $\theta = \frac{x_0-\mu}{\sigma}$. We start by rewriting the equation so that the right side equals 0:

$$G_{n-1,\sqrt{n}B_L}\left(\frac{x_0-\bar{x}}{s/\sqrt{n}}\right)+\alpha-1=0. \tag{16}$$

In equation 16, n, x_0, $\bar{x}$, s, and α are fixed constants, and the only unknown is B_L. Therefore, if we let $H(B_L;n,x_0,\bar{x},s,\alpha)=G_{n-1,\sqrt{n}B_L}\left(\frac{x_0-\bar{x}}{s/\sqrt{n}}\right)+\alpha-1$, the problem becomes one of solving for the root of the function H. We can show H to be a monotonic decreasing function in B_L by recognizing that the non-central t distribution function is monotonic decreasing in the non-centrality parameter (see figure 1) or by formal proof (see appendix A). Therefore, one can use the bisection method or other root-finding algorithm to solve for B_L. Finally, the LCB for the probability of non-perforation is $\Phi(B_L)$.

The bisection method can be easily programmed into most mathematical software packages. MATLAB[5] and Mathematica[6] programs appear in appendices B through G for calculating an LCB, a UCB, and a two-sided confidence interval for $\Phi\left(\frac{x_0-\mu}{\sigma}\right)$.

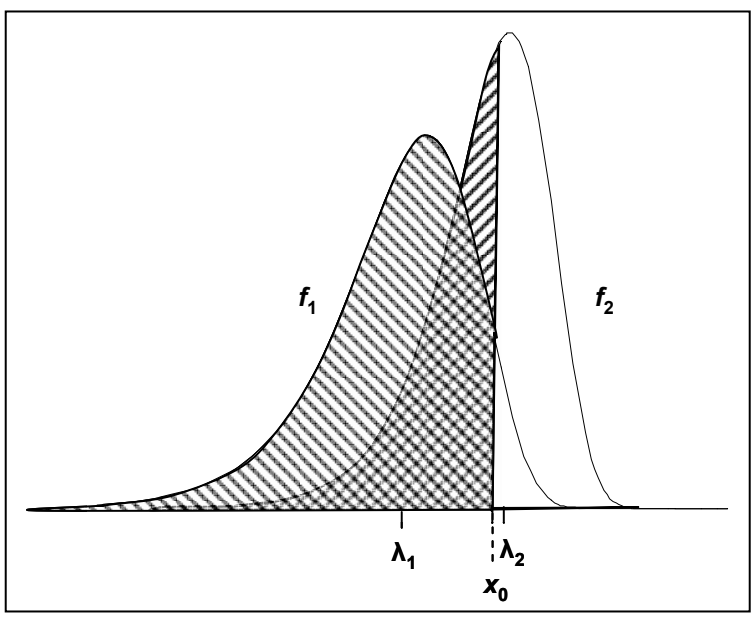

Figure 1. Two non-central t density functions, f_1 and f_2, with respective non-centrality parameters λ_1 and λ_2. (As the non-centrality parameter increases, the area to the right of x_0 under the non-central t curve decreases.)

[5]MATLAB is a registered trademark of The MathWorks.

[6]Mathematica is a registered trademark of Wolfram Research, Inc.

7. Confidence Bands

By plotting the function $\Phi\left(\frac{x-\bar{x}}{s}\right)$ for all real numbers *x*, one obtains a smooth, monotonically increasing function that is a simple estimate for the entire CDF. A lower confidence band for the CDF is a plot of the function $\Phi(B_L(x))$. This band lies below the estimated CDF and enables one to see the region above the band in which the entire actual CDF lies (with the stated level of confidence). If one fixes the probability level on the vertical axis, then the corresponding x-axis value from the lower confidence band is an upper tolerance bound. In the context of armor design, the upper tolerance bound is very important as it states with a specified level of confidence what armor thickness is needed to stop a (high) percentage of projectiles from perforating the materiel. A MATLAB program for generating a lower confidence band appears in appendix H.

Upper confidence bands and two-sided confidence bands are constructed in a similar fashion; however, they are likely to be of much less interest to the armor designer. Although MATLAB programs for their construction are not included here, the interested reader could easily tailor the program in appendix H to accomplish this.

8. An Application

Table 1 gives the penetrations of 14 projectiles into an extended armor pack. Engineers plan to use 115 units as the armor thickness. To determine if this thickness will provide enough protection, an estimate is desired for the probability that the next projectile fired will not penetrate deeper than 115 units. Letting X be the penetration of this next projectile, we seek an estimate for $P(X \le 115)$.

Table 1. Penetration depths into armor (no units specified).

47	81	89	99	114
59	86	90	100	118
80	88	90	113	

The data are first checked for normality by Lilliefors Test (see Conover, 1980). The value of the test statistic is 0.1713, which corresponds to a P-value greater than 0.20. Therefore, the assumption of normality is not rejected.

Summary statistics for the sample data are $\bar{x} = 89.571$ and $s^2 = 390.725$. Equation 2 is used to obtain a point estimate for $P(X \le 115)$. The solution is

$$\hat{P}(X \le 115) = \Phi\left(\frac{115 - 89.571}{\sqrt{390.725}}\right) = \Phi(1.286) = 0.901 .$$

To achieve a 95% LCB for $P(X \le 115)$, we start by using equation 12:

$$.95 = G_{13,\sqrt{14}B_L}(4.813) .$$

Using the bisection method, we obtain a solution of $B_L = 0.6672$. Therefore, a 95% LCB for $P(X \le 115)$ is $\Phi(0.6672) = .7477$. That is, for an armor of thickness 115 units, one can be 95% confident that the probability of non-perforation is at least 74.77%.

At this point, a reasonable question might be "With 95% confidence, what armor thickness would offer at least 90% protection against perforation?" The lower confidence band for $\Phi(B_L)$, shown in figure 2, will help determine this. We begin by extending a horizontal (dashed) line from the "y"-axis at $P(non - perforation) = .90$ to the lower confidence band and then dropping to the x-axis to find the desired armor thickness. This thickness of 131.25 is the 90% upper tolerance bound.

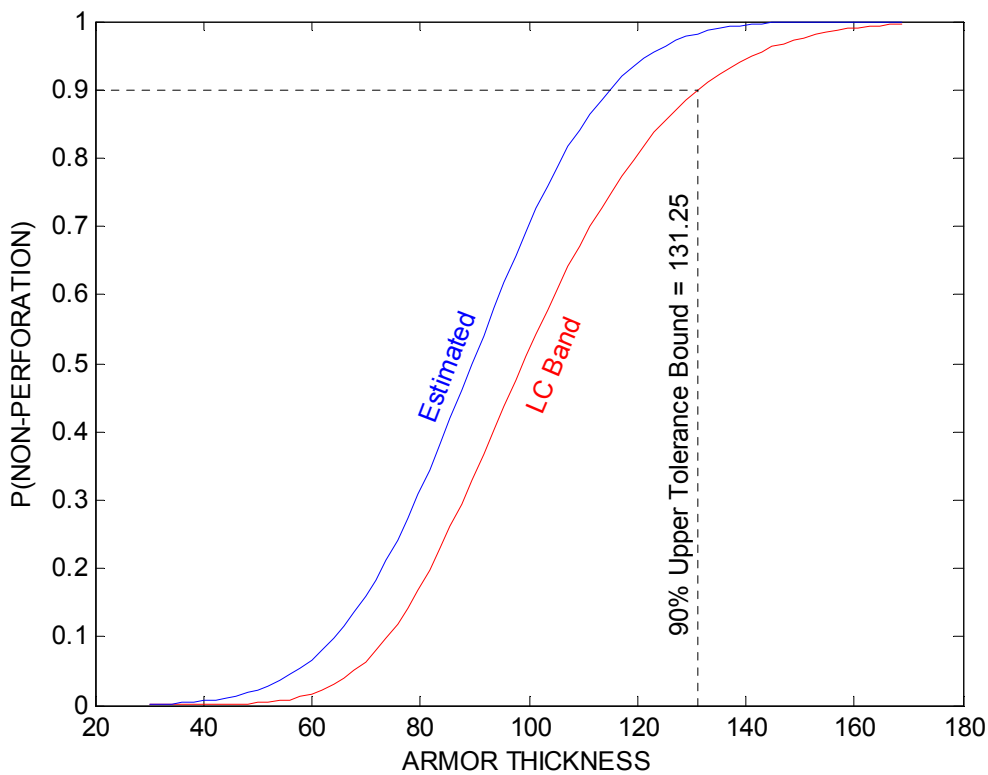

Figure 2. Estimated probability of non-perforation and 95% LCB as a function of armor thickness. (The line formed by the bounds is referred to as the lower confidence band. From the band, one is able to determine what armor thickness yields a certain minimum probability of non-perforation, e.g., with 95% confidence, one can state that a thickness of 131.25 will stop at least 90% of projectiles from perforating the material.)

A 95% two-sided confidence interval for $P(X \le 115)$ is calculated from equations 12 and 13. It has a lower limit given by $\Phi(C_L)$ in which C_L is the solution to $1-\frac{\alpha}{2} = G_{n-1,\sqrt{n}C_L}\left(\frac{x_0-\bar{x}}{s/\sqrt{n}}\right)$, or $.975 = G_{13,\sqrt{14}C_L}(4.813)$. The upper limit is given by $\Phi(C_U)$ in which C_U is the solution to $\frac{\alpha}{2} = G_{n-1,\sqrt{n}C_U}\left(\frac{x_0-\bar{x}}{s/\sqrt{n}}\right)$, or $.025 = G_{13,\sqrt{14}C_U}(4.813)$. Using the bisection method again, we obtain $C_L = .5566$ and $C_U = 1.9901$. Therefore, the lower confidence limit is $\Phi(.5566) = .7111$, and the upper confidence limit is $\Phi(1.9901) = .9767$. We can be 95% confident that the true probability of non-perforation is between 71.11% and 97.67%.

9. Summary

Reporting an interval estimate along with a point estimate gives designers a sense of the error of estimation which is a function of the sample size used in the study and the inherent variability of each projectile-armor interaction. Using a generalized pivotal quantity, we have achieved an interval estimate for the probability that a homogeneous armor plate of specified thickness will successfully stop a projectile from perforating. In the derivation of this interval estimate, we have assumed that the penetration depths are random and that they follow a normal distribution of unknown mean and variance.

The interval limit (or limits if a two-sided interval is desired) is a function of the sample mean and sample standard deviation of penetration depths, the number of shots fired, the thickness of the armor and percentiles of a non-central t distribution. However, since the non-centrality parameter associated with this distribution is a function of the interval limit, numerical methods are required to obtain the final solution. MATLAB and Mathematica codes to perform these calculations are provided in appendices B through G.

Confidence bands that graphically display the entire the relationship between armor width and bounds on the probability of non-perforation follow naturally. These bands may be of help to armor designers in selecting an armor width that will provide a high degree of protection against specific enemy threats.

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Appendix A. Proof of the Monotonicity of the Function H

Show that $H(\cdot)$ is a monotonic decreasing function. That is, show that if $\lambda_1 < \lambda_2$, then $H(\lambda_1) > H(\lambda_2)$.

Proof:

Let n, x_0, $\bar{x}$, s, α, λ_1, and λ_2 be fixed constants with $\lambda_1 < \lambda_2$. Let X_1 and Y be independent random variables, X_1 being normally distributed with mean λ_1 and variance 1, and Y following a chi-squared distribution with $n-1$ degrees of freedom. The support of X_1 is the set of all real numbers, and the support of Y is the set of all positive reals.

Now define $W = \sqrt{\frac{nY}{n-1}}\frac{x_0 - \bar{x}}{s}$. The support of W is either the set of all positive real numbers (if $x_0 \geq \bar{x}$) or the set of all non-positive real numbers (if $x_0 < \bar{x}$). However, in general, we denote the support of W by Θ.

$P(X_1 < W)$ can be expressed as a double integral of the joint density of X_1 and W, namely,

$$P(X_1 < W) = \int_{\Theta}\int_{-\infty}^{w} f_{X_1}(x) f_W(w) dx\, dw. \tag{A-1}$$

The double integral of equation A-1, can be rewritten as the sum of two double integrals:

$$P(X_1 < W) = \int_{\Theta}\int_{-\infty}^{w-(\lambda_2-\lambda_1)} f_{X_1}(x) f_W(w) dx\, dw + \int_{\Theta}\int_{w-(\lambda_2-\lambda_1)}^{w} f_{X_1}(x) f_W(w) dx\, dw. \tag{A-2}$$

Each of these double integrals is positive, so by dropping the latter of the two, we obtain the inequality,

$$P(X_1 < W) > \int_{\Theta}\int_{-\infty}^{w-(\lambda_2-\lambda_1)} f_{X_1}(x) f_W(w) dx\, dw. \tag{A-3}$$

However, the double integral in equation A-3, represents a probability, namely, $P(X_1 < W - (\lambda_2 - \lambda_1))$, which in turn equals $P(X_1 + \lambda_2 - \lambda_1 < W)$. So,

$$P(X_1 < W) > P(X_1 + \lambda_2 - \lambda_1 < W). \tag{A-4}$$

Now define $X_2 = X_1 + (\lambda_2 - \lambda_1)$; then X_2 is normally distributed with mean λ_2 and variance 1. Therefore,

$$P(X_1 < W) > P(X_2 < W), \tag{A-5}$$

Consider the standard normal random variable, Z, having mean 0 and variance 1. Employing the equalities $X_1 = Z + \lambda_1$ and $X_2 = Z + \lambda_2$ and the definition of *W*, we expand the above result to obtain

$$P\left(Z + \lambda_1 < \sqrt{\frac{nY}{n-1}}\,\frac{x_0 - \bar{x}}{s}\right) > P\left(Z + \lambda_2 < \sqrt{\frac{nY}{n-1}}\,\frac{x_0 - \bar{x}}{s}\right). \tag{A-6}$$

Rearranging the expressions inside each of the probability statements,

$$P\left(\frac{Z + \lambda_1}{\sqrt{Y/n-1}} < \frac{x_0 - \bar{x}}{s/\sqrt{n}}\right) > P\left(\frac{Z + \lambda_2}{\sqrt{Y/n-1}} < \frac{x_0 - \bar{x}}{s/\sqrt{n}}\right). \tag{A-7}$$

By the definition of a non-central t random variable,

$$P\left(T_{n-1,\lambda_1} < \frac{x_0 - \bar{x}}{s/\sqrt{n}}\right) > P\left(T_{n-1,\lambda_2} < \frac{x_0 - \bar{x}}{s/\sqrt{n}}\right). \tag{A-8}$$

Each of these probabilities is the cumulative non-central t distribution function evaluated at $\frac{x_0 - \bar{x}}{s/\sqrt{n}}$:

$$G_{n-1,\lambda_1}\left(\frac{x_0 - \bar{x}}{s/\sqrt{n}}\right) > G_{n-1,\lambda_2}\left(\frac{x_0 - \bar{x}}{s/\sqrt{n}}\right). \tag{A-9}$$

Next, we add the constant $\alpha - 1$ to both sides of the inequality:

$$G_{n-1,\lambda_1}\left(\frac{x_0 - \bar{x}}{s/\sqrt{n}}\right) + \alpha - 1 > G_{n-1,\lambda_2}\left(\frac{x_0 - \bar{x}}{s/\sqrt{n}}\right) + \alpha - 1. \tag{A-10}$$

Using the definition of the function $H(\cdot)$, we obtain the desired result.

$$H(\lambda_1) > H(\lambda_2). \tag{A-11}$$

Appendix B. MATLAB Code for Probability of Non-perforation LCB

MATLAB Release 14, Version 7, code for calculating an LCB for the probability of non-perforation, $\Phi\left(\frac{x_0-\mu}{\sigma}\right)$. The user may supply his or her own data and values for alpha, x0, and epsilon.

```
% Define variables %
alpha=.05;   x0=115;   epsilon=.000001;

% Declare data %
data=[47 59 80 81 86 88 89 90 90 99 100 113 114 118]';

% Calculate summary statistics and define K %
n=length(data);   xbar=mean(data);   sigma=std(data);
K=sqrt(n)*(x0- xbar)/sigma;

% Initialize flag and bounds on non-centrality parameter %
flag=0;   lononcent=xbar-10*sigma;   hinoncent=xbar+10*sigma;

% Execute bisection method %
noncent=(lononcent+hinoncent)/2;
while flag==0
    if hinoncent-lononcent < epsilon
        flag=1;
    elseif nctcdf(K,n-1,noncent)+alpha-1 < 0
        hinoncent=noncent;  noncent=(noncent+lononcent)/2;
    else
        lononcent=noncent;  noncent=(noncent+hinoncent)/2;
    end
end

%Print lower confidence bound for probability of non-perforation %
fprintf('%3.0f%% LCB for P(non-perforation) = %6.4f\n',...
    100*(1-alpha),normcdf(noncent/sqrt(n)))
```

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Appendix C. Mathematica Code for Probability of Non-perforation LCB

Mathematica Version 5.1 code for calculating an LCB for the probability of non-perforation, $\Phi\left(\frac{x_0-\mu}{\sigma}\right)$. The user may supply his or her own data and values for alpha, x0, and epsilon.

```
(* Remove warning messages that do not affect results but clutter output *)
Off[General::spell]; Off[General::spell1];
Off[NIntegrate::slwcon]; Off[NIntegrate::ncvb];

(* Enable use of statistical tools *)
<<Statistics`ContinuousDistributions`
<<Statistics`NormalDistribution`

(* Clear variables for use *)
Clear[data,n,noncent,x0,a,epsilon,lononcent,hinoncent,flag,K,ndist]

(* Define variables *)
alpha=.05; x0=115; epsilon =.000001;

(* Declare data *)
data={47,59,80,81,86,88,89,90,90,99,100,113,114,118};

(* Calculate summary statistics and define K *)
n=Length[data];  xbar=Mean[data]//N;  sigma=StandardDeviation[data]//N;
K=(x0-xbar)/(sigma/Sqrt[n])//N;

(* Initialize flag and bounds on non-centrality parameter *)
flag=0; lononcent=xbar-10sigma//N; hinoncent=xbar+10sigma//N;

(* Execute bisection method *)
H:=CDF[NoncentralStudentTDistribution[n-1,noncent],K]+alpha-1
While[flag==0,noncent=Mean[{lononcent,hinoncent}];
      If[hinoncent-lononcent<epsilon,flag=1,
      If[H<0,hinoncent=noncent,lononcent=noncent]]]

(* Print lower confidence bound for probability of non-perforation *)
Print[TraditionalForm[StringForm["``0% LCB for P(non-perforation) = ``",
      100 (1-alpha),CDF[NormalDistribution[0,1],noncent/Sqrt[n]]]]]
```

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Appendix D. MATLAB Code for Probability of Non-perforation UCB

MATLAB Release 14, Version 7, code for calculating a UCB for the probability of non-perforation, $\Phi\left(\frac{x_0-\mu}{\sigma}\right)$. The user may supply his or her own data and values for alpha, x0, and epsilon.

```
% Define variables %
alpha=.05;   x0=115;   epsilon=.000001;

% Declare data %
data=[47 59 80 81 86 88 89 90 90 99 100 113 114 118]';

% Calculate summary statistics and define K %
n=length(data);   xbar=mean(data);   sigma=std(data);
K=sqrt(n)*(x0- xbar)/sigma;

% Initialize flag and bounds on non-centrality parameter %
flag=0;   lononcent=xbar-10*sigma;   hinoncent=xbar+10*sigma;

% Execute bisection method %
noncent=(lononcent+hinoncent)/2;
while flag==0
    if hinoncent-lononcent < epsilon
        flag=1;
    elseif nctcdf(K,n-1,noncent)-alpha < 0
        hinoncent=noncent;  noncent=(noncent+lononcent)/2;
    else
        lononcent=noncent;  noncent=(noncent+hinoncent)/2;
    end
end

% Print upper confidence bound for probability of non-perforation %
fprintf('%3.0f%% UCB for P(non-perforation) = %6.4f\n',...
    100*(1-alpha),normcdf(noncent/sqrt(n)))
```

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Appendix E. Mathematica Code for Probability of Non-perforation UCB

Mathematica Version 5.1 code for calculating a UCB for the probability of non-perforation, $\Phi\left(\frac{x_0-\mu}{\sigma}\right)$. The user may supply his or her own data and values for alpha, x0, and epsilon.

```
(* Remove warning messages that do not affect results but clutter output *)
Off[General::spell]; Off[General::spell1];
Off[NIntegrate::slwcon]; Off[NIntegrate::ncvb];

(* Enable use of statistical tools *)
<<Statistics`ContinuousDistributions`
<<Statistics`NormalDistribution`

(* Clear variables for use *)
Clear[data,n,noncent,x0,a,epsilon,lononcent,hinoncent,flag,K,ndist]

(* Define variables *)
alpha=.05; x0=115; epsilon =.000001;

(* Declare data *)
data={47,59,80,81,86,88,89,90,90,99,100,113,114,118};

(* Calculate summary statistics and define K *)
n=Length[data];  xbar=Mean[data]//N;  sigma=StandardDeviation[data]//N;
K=(x0-xbar)/(sigma/Sqrt[n])//N;

(* Initialize flag and bounds on non-centrality parameter *)
flag=0; lononcent=xbar-10sigma//N; hinoncent=xbar+10sigma//N;

(* Execute bisection method *)
H:=CDF[NoncentralStudentTDistribution[n-1,noncent],K]-alpha
While[flag==0,noncent=Mean[{lononcent,hinoncent}];
      If[hinoncent-lononcent<epsilon,flag=1,
      If[H<0,hinoncent=noncent,lononcent=noncent]]]

(* Print upper confidence bound for probability of non-perforation *)
Print[TraditionalForm[StringForm["``0% UCB for P(non-perforation) = ``",
      100 (1-alpha),CDF[NormalDistribution[0,1],noncent/Sqrt[n]]]]]
```

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Appendix F. MATLAB Code for Probability of Perforation Two-Sided Confidence Interval

MATLAB Release 14, Version 7, code for calculating a two-sided confidence interval for the probability of non-perforation, $\Phi\left(\frac{x_0-\mu}{\sigma}\right)$. The user may supply his or her own data and values for alpha, x0, and epsilon.

```
% Define variables %
alpha=.05;   x0=115;   epsilon=.000001;

% Declare data %
data=[47 59 80 81 86 88 89 90 90 99 100 113 114 118]';

% Calculate summary statistics and define K %
n=length(data);   xbar=mean(data);   sigma=std(data);
K=sqrt(n)*(x0- xbar)/sigma;

% Initialize flag and bounds on non-centrality parameter %
flag=0;   lononcent=xbar-10*sigma;   hinoncent= xbar+10*sigma;

% Execute bisection method for lower limit %
noncent=(lononcent+hinoncent)/2;
while flag==0
    if hinoncent-lononcent < epsilon
        flag=1;
    elseif nctcdf(K,n-1,noncent)+alpha/2-1 < 0
        hinoncent=noncent;  noncent=(noncent+lononcent)/2;
    else
        lononcent=noncent;  noncent=(noncent+hinoncent)/2;
    end
end

% Print lower confidence limit for probability of non-perforation %
fprintf('%4.0f%% CI for P(non-perforation) = (%6.4f, ',...
    100*(1-alpha),normcdf(noncent/sqrt(n)))

% Re-initialize flag and bounds on non-centrality parameter %
flag=0;   lononcent=xbar-10*sigma;   hinoncent= xbar+10*sigma;
```

```
% Execute bisection method for upper limit %
noncent=(lononcent+hinoncent)/2;
while flag==0
    if hinoncent-lononcent < epsilon
        flag=1;
    elseif nctcdf(K,n-1,noncent)-alpha/2 < 0
        hinoncent=noncent;  noncent=(noncent+lononcent)/2;
    else
        lononcent=noncent;  noncent=(noncent+hinoncent)/2;
    end
end

% Print upper confidence limit for probability of non-perforation %
fprintf('%6.4f)\n',normcdf(noncent/sqrt(n)))
```

Appendix G. Mathematica Code for Probability of Perforation Two-Sided Confidence Interval

Mathematica Version 5.1 code for calculating a two-sided confidence interval for the probability of non-perforation, $\Phi\left(\frac{x_0-\mu}{\sigma}\right)$. The user may supply his or her own data and values for alpha, x0, and epsilon.

```
(* Remove warning messages that do not affect results but clutter output *)
Off[General::spell]; Off[General::spell1];
Off[NIntegrate::slwcon]; Off[NIntegrate::ncvb];

(* Enable use of statistical tools *)
<<Statistics`ContinuousDistributions`
<<Statistics`NormalDistribution`

(* Clear variables for use *)
Clear[data,n,noncent,x0,a,epsilon,lononcent,hinoncent,flag,K,
      ndist,lowerlimit,upperlimit]

(* Define variables *)
alpha=.05; x0=115; epsilon =.000001;

(* Declare data *)
data={47,59,80,81,86,88,89,90,90,99,100,113,114,118};

(* Calculate summary statistics and define K *)
n=Length[data];  xbar=Mean[data]//N;  sigma=StandardDeviation[data]//N;
K=(x0-xbar)/(sigma/Sqrt[n])//N;

(* Initialize flag and bounds on non-centrality parameter *)
flag=0; lononcent=xbar-10sigma//N; hinoncent=xbar+10sigma//N;

(* Execute bisection method for lower limit *)
H:=CDF[NoncentralStudentTDistribution[n-1,noncent],K]+alpha/2-1
While[flag==0,noncent=Mean[{lononcent,hinoncent}];
      If[hinoncent-lononcent<epsilon,flag=1,
```

```
      If[H<0,hinoncent=noncent,lononcent=noncent]]]
lowerlimit = N[CDF[NormalDistribution[0,1],noncent/Sqrt[n]]];

(* Re-initialize flag and bounds on non-centrality parameter *)
flag=0; lononcent=xbar-10sigma//N; hinoncent=xbar+10sigma//N;

(* Execute bisection method for upper limit *)
H:=CDF[NoncentralStudentTDistribution[n-1,noncent],K]-alpha/2
While[flag==0,noncent=Mean[{lononcent,hinoncent}];
      If[hinoncent-lononcent<epsilon,flag=1,
      If[H<0,hinoncent=noncent,lononcent=noncent]]]
upperlimit = N[CDF[NormalDistribution[0,1],noncent/Sqrt[n]]];

(* Print confidence interval for probability of non-perforation *)
Print[TraditionalForm[StringForm["``0% CI for P(non-perforation) = (``,``)",
      100 (1-alpha),lowerlimit,upperlimit]]]
```

Appendix H. MATLAB Code for Probability of Non-perforation CDF and Lower Confidence Band

MATLAB Release 14, Version 7, code for plotting an estimated CDF and lower confidence band. The user may supply his or her own data and values for alpha, x0, and epsilon.

```
% Define variables %
alpha=.05;   epsilon=.0001;

% Declare data %
data=[47 59 80 81 86 88 89 90 90 99 100 113 114 118]';

% Calculate summary statistics %
n=length(data);  Xbar=mean(data);  SDev=std(data);

% Initialize counter for points along the X-axis %
i=0;

% Declare armor thickness %
for x0=Xbar-3*SDev:SDev/10:Xbar+4*SDev
    i=i+1; x(i)=x0;

    % Calculate plug-in estimate for probability of non-perforation %
    pointest(i)=normcdf((x0-Xbar)/SDev);

    % Define K %
    K=sqrt(n)*(x0-Xbar)/SDev;

    % Initialize flag and bounds on non-centrality parameter %
    flag=0;   lononcent=-1000;   hinoncent=1000;

    % Execute bisection method for lower confidence bound %
    noncent=(lononcent+hinoncent)/2;
    while flag==0
        if hinoncent-lononcent < epsilon
            flag=1;
        elseif nctcdf(K,n-1,noncent)-1+alpha < 0
```

```
            hinoncent=noncent;  noncent=(noncent+lononcent)/2;
        else
            lononcent=noncent;  noncent=(noncent+hinoncent)/2;
        end
    end
    lcb(i)=normcdf(noncent/sqrt(n));
end

% as a function of armor thickness, plot non-perforation probability curve %
% in blue, and lower confidence band for probability of non-perforation in %
% red %
plot(x,pointest,'b-',x,lcb,'r-')
xlabel('ARMOR THICKNESS'); ylabel('P(NON-PERFORATION)')
text(Xbar-.25*SDev,.5,'Estimated','Color','b','Rotation',69,...
    'HorizontalAlignment','Center')
text(Xbar+.75*SDev,.5,'LC Band','Color','r','Rotation',67,...
    'HorizontalAlignment','Center')
```

List of Symbols Used

Scalar Values

n	number of observations in sample data
r_i	the ith observed value of the random variable R
$r_{(i)}$	the ith ordered, observed value of the random variable R
s	observed sample standard deviation
x_i	the ith observed value from a data set; in this paper, the observed depth of penetration from the ith projectile
x_0	a specified thickness of armor
$\bar{x}$	observed sample mean
z_p	the value from the standard normal distribution having an area of p to its right under the density curve
α	level of significance
σ	population standard deviation for a normally distributed random variable
θ	parameter for which a confidence interval is constructed; may actually be a function of one or more parameters
μ	population mean for a normally distributed random variable
ξ	a set of one or more nuisance parameters

Random Variables

$R(\ )$	generalized test variable; a function of random data, observed data, parameter of interest and perhaps other nuisance parameters
S	sample standard deviation of depths of penetration
T	student's t random variable
V	chi-square random variable
X	depth of penetration for a randomly selected projectile
$\bar{X}$	sample mean depth of penetration

Y a normal random variable with mean zero and variance $\frac{1}{n}$

Z a standard normal random variable, with mean zero and variance one

Functions

$F(\ )$ cumulative distribution function

$\hat{F}(\ )$ estimated cumulative distribution function

$G(\ )$ student's t cumulative distribution function

$\Phi(\ )$ standard normal cumulative distribution function

Miscellaneous

B_L lower confidence bound

B_U upper confidence bound

C_L lower confidence limit

C_U upper confidence limit

$P(\ )$ probability of the parenthesized expression

$\hat{P}(\ )$ estimated probability of the parenthesized expression

Σ summation

^ estimate of

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