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Comparisons of Analytical and Numerical Calculations of Communications Probability

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COMPARISONS OF ANALYTICAL AND NUMERICAL CALCULATIONS OF COMMUNICATIONS PROBABILITY

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Middleton's analytical formula for the probability of communications in a land mobile environment and Berry's computer program for probabilistic calculations of interference are shown to produce identical answers for identical inputs. The assumptions, input, and output of the two models are compared. Middleton's assumption that interfering sources transmit randomly and independently is of particular interest for mobile radio services. The effects of the greater variety of input assumptions possible with the computer method are illustrated numerically.

To use Middleton's formula to compute the probability of interference for a detailed scenario, the parameters of Class A noise must be determined from the scenario descriptors. Methods for finding these parameters are explored.

To illustrate possible applications, the number of statistically identical links that can operate with specified reliability in a given area is computed with the numerical model.

Key words: Class A noise; communications probability; mobile radio; probabilistic EMC; probability of interference

1. INTRODUCTION

Two models have recently been developed for computing the probability of communications in a congested spectral-use environment. Middleton has derived an analytical formula--an infinite series of integrals--for the probability that the signal-to-noise ratio (SNR) exceeds a specified threshold (Middleton, 1979a). He calls this probability the _probability of noninterference_ (PNI) and denotes its value by P_a. Berry had developed a computer program to compute this same probability (Berry, 1977). Successful frequency management depends in part on knowledge of the probability of interference, so these models are potentially valuable tools for frequency managers. The purpose of this report is to make both models immediately accessible to frequency managers. The modeling assumptions, required input data, and output are compared to help the user decide when each method is appropriate and capable.

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1.1 General Comparison of Analytical Solutions and Numerical Methods

In this report, an analytical solution is an equation of the desired quantity and an algebraic combination of specific mathematical functions. An analytic solution does not include indicated derivatives or integrations, nor purely symbolic functions whose form is yet to be determined.

Analytical solutions are distinguished from formal solutions in which the expression for the desired quantity may contain symbolic operators (such as integrals or derivatives) and general functions whose form has not yet been specified. A formal solution is essentially a plan for deriving an analytical solution or for getting numerical results. To get the results, one must specify the form of the general functions and carry out the indicated operations analytically (if possible) or numerically.

All other things being equal, analytical solutions are preferable to computer programs for at least two reasons. First, the analytical solution provides a direct, explicit relationship between the input variables and the desired answer. If the solution is simple enough, the effect of varying different parameters of the problem can be perceived just by studying the relationship. The behavior of the answer for limiting values of the parameters can be derived. Second, if the solution is complicated enough that numerical evaluation is required, the programming is usually straightforward, requires only a medium-to-small computer, and program execution uses relatively little computer time.

Unfortunately, analytical solutions have not yet been derived for many practical engineering problems, including some for which formal solutions exist. In these cases numerical evaluation of the formal solution can provide needed results immediately. Computer programs can also evaluate formal solutions for general cases that will never yield to analytical solution. The disadvantages of the numerical approach are the requirement for greater computer power, longer running time, and lack of direct insight into the behavior of the solution because the relationships are implicit rather than explicit.

1.2 Overview of This Report

These general differences between analytical solutions and numerical evaluations of formal solutions are illustrated by the comparisons in this report. Both the computer program and the analytical solution are based on quite general formal solutions for the probability of interference in a congested environment. In their most general forms, these two formal solutions are essentially similar and of little immediate use to frequency managers, who do not have time to program computers nor to carry out difficult derivations. Therefore, this report compares only the available analytical solution (Middleton, 1979a) with the available computer program (Berry, 1978).

In the next section of this report, the analytical solution derived by Middleton (1979a) and the numerical evaluation are compared. The differences between the approaches and the input required and allowed are explored. Numerical results for identical input are compared and found to agree. This agreement lends some credence to both methods. It is unlikely that agreement between two such different approaches is merely fortuitous.

To get an analytical solution, Middleton (1979a) idealized some features of the problem and restricted the scenario to certain specialized cases. The computer program can accept a greater range and variety of scenarios as input. So, in Section 3, the computer program is used to calculate the changes in the probability of interference for different scenarios. These differences illuminate the effects of Middleton's assumptions.

In Middleton's (1979a) formula for PNI, the noise and interference environment is described by the three parameters of Class A noise.[1] In a practical application, one is more likely to know the powers, locations, and traffic patterns of interfering transmitters. So to use the formulas, these known parameters of the scenario must be related to the Class A noise parameters. Physical interpretation of the terms in the series for Class A noise and reference to the formal solution provide some insight, but no satisfactory analytical solution for strictly canonical Class A noise parameters as functions of scenario descriptors could be found. (Such a direct calculation is possible with a newly developed, quasi-canonical Class A model [Middleton, private communication, 1980].) It is apparently necessary to compute Class A noise parameters numerically using a program similar to Berry's. Section 4 of this report describes the search for a connection between a scenario and Class A noise and illustrates the hybrid calculation that is apparently necessary in a practical application.

2. COMPARISONS OF THE TWO MODELS

Middleton's (1979a) formula for the probability that the signal-to-noise ratio will be satisfactory is an extension of his development of canonical noise models (e.g., Middleton, 1972, 1976). To develop these models, he began with quite general situations and fundamental equations, then added restrictions and approximations as

[1]A similar formula for Class B noise will soon be published (private communication from D. Middleton, 1980).

necessary to obtain a tractable result. Each restriction or approximation was carefully incorporated into the notation, so that there would be no confusion between the general beginnings and the somewhat specialized results. Although this practice ensured clarity, it produced a cumbersome notation. This report deals only with the final solution for one class of noise, so a more simplified notation is used with the restrictions treated as given. Each symbol will be defined when it is introduced, and Table 1 defines all symbols and shows their relationship to Middleton's notation.

2.1 Middleton's Analytical Solution for Land Mobile Radio Communications

Middleton's formal solution for P_a, the probability that the SNR exceeds the required threshold, is (Middleton, 1979a)

$$P_a = P(x \geq r) = \int_{x_0}^{\infty} \int_{0}^{\infty} z\, w_S(xz|\theta_S)\, w_I(z|\theta_I)\, dz\, dx \quad , \tag{1}$$

where

x is the signal-to-noise ratio,

r is the required SNR threshold,

$w_S(s|\theta_S)$ is the probability density function (pdf) of the signal, given the the input parameters, θ_S, that influence the signal, and

$w_I(z|\theta_I)$ is the pdf of interference given the input parameters, θ_I, that influence the interference.

To actually evaluate (1), the pdf's must be specified in computable form, and the indicated double integration must be performed.

Middleton (1979a) assumes that the interference environment is represented by a Class A noise model. The probability density function (pdf) of the noise power, measured after the IF filter in the receiver, is

$$w_I(i_n) = e^{-A} \sum_{m=0}^{\infty} \frac{A^m}{m!} \frac{g_m}{\bar{i}_n} \exp(-i_n g_m/\bar{i}_n) \quad , \tag{2}$$

where i_n is the total (instantaneous) received noise power in watts, and

$$g_m = \frac{A(1 + \Gamma)}{m + A\Gamma} \quad . \tag{3}$$

In (3),

$$\Gamma = \bar{n}/\bar{i}_c \tag{4}$$

is the ratio between the mean gaussian noise, $\bar{n}$, and the mean nongaussian noise, $\bar{i}_c$. (Throughout this report a bar over a variable will denote the mean of the

Table 1. Notation Used in This Report Related to Notation Used in Middleton (1979a)

This Report			Middleton
Symbol	Units	Notes	Symbol
A	*	$A = \nu_R \overline{T}_R$	A_A
d	km	Distance from transmitter to receiver	r_D
d_a	km	Distance from receiver to transmitter's center of operations	r_o
d_d	*	Normalized distance, $d_d = d/d_o$	r_d
d_o	km	A reference distance in path length pdf	$\hat{r}_o$
D	*	$D = 0.1 \log_e 10$	---
e_o	volts	Limiting sensitivity voltage of receiver	$e_{o\gamma}^{(A)}$
g_m	*	$g_m = A(1 + \Gamma)/(m + A\Gamma)$	$(2\sigma_{mA}^2)^{-1}$
Γ	*	$\Gamma = \overline{n}/\overline{i}_c$	Γ'_A
γ	*	Exponent in propagation law, decay rate	γ
i_n	watts	Total noise power, $i_n = n + i_c$	I_N
$\overline{i}_n$	watts	Mean of i_n, $\overline{i}_n = \overline{i}_c(1 + \Gamma)$	$\overline{I}_N$
I_N	dBW	$I_N = 10 \log_{10} i_n$	---
i_c	watts	Nongaussian noise, "interference"	---
I_C	dBW	$I_C = 10 \log_{10} i_c$	---
$\overline{i}_c$	watts	Mean of i_c	Ω_{2A}
n	watts	Gaussian noise	---
$\overline{n}$	watts	Mean of n	σ_G^2
N	dBW	$N = 10 \log_{10} n$	---
ν	s^{-1}	Average number of emissions by interfering sources, per second	---
ν_R	s^{-1}	Average number of interfering emissions measured after IF filter, $\nu_R \le \nu$	ν_∞

Table 1. Notation Used in This Report Related to Notation Used in Middleton (1979a) (continued)

This Report			Middleton
Symbol	Units	Notes	Symbol
P_a	*	$P(s/i_n \geq r) = P(S-I_N \geq R)$	P_a
P_m	*	An integral, see equation (12)	P_m
p_1	watts	Signal power received at 1 km; defines power of transmitter	I_{oS}
r	*	Required ("threshold") signal-to-noise ratio	$\overline{x}_o$
r_{oa}	*	$r_{oa} = d_a/d_o$	r_{oa}
R	dB	$R = 10 \log_{10} r$	---
s	watts	Wanted signal power in receiver	I_S
S	dBW	$S = 10 \log_{10} s$	---
σ	km	$= d_o/\sqrt{2}$	$\hat{\sigma}_o$
$\overline{T}$	s	Average length of an emission from an interference source	---
$\overline{T}_R$	s	Average length of an interfering emission measured after the IF filter, $\overline{T}_R \leq \overline{T}$	$\overline{T}_S$
U	*	Traffic intensity, $U = \nu\overline{T} \geq A$	---
y	*	$\hat{x}_o/\overline{x}_o = p_1/(\overline{i}_n r)$	$\hat{x}_o/\overline{x}_o$

variable.) The total noise, i_n (which includes both "noise" and "interference"), is the sum of background "noise," n, which is gaussian in nature, and structured or impulsive "interference," i_c. That is

$$i_n = i_c + n \quad , \tag{5}$$

and

$$\bar{i}_n = \bar{i}_c + \bar{n} = \bar{i}_c(1 + \Gamma) \quad . \tag{6}$$

The "overlap index," A, is

$$A = \nu_R \bar{T}_R \quad , \tag{7}$$

where ν_R is the average number of interfering emissions per second measured in the receiver after the intermediate frequency (IF) filter, and $\bar{T}_R$ is the average length in seconds of those measured emissions. In the derivation of (2), Middleton assumes that emissions from interfering sources begin at random times and are independent.

Middleton assumes that the parameters of the Class A noise environment have been determined by measurement and has described methods for making the measurements (Middleton, 1979b). Adequate measurement programs will take considerable time and money, and some frequency planning problems include systems that are not yet built. It will sometimes be necessary to estimate the Class A noise parameters from a description of the systems and their deployment. This problem is discussed in more detail in Section 4. In this section, it is assumed that the parameters A, Γ, and $\bar{i}_n$ are known.

The propagation law assumed by Middleton results in a received signal,

$$s = \frac{p_1}{d^{2\gamma}} \quad , \quad \text{watts} \quad , \tag{8}$$

where p_1 represents the effective radiated power of the transmitter by giving the received power at a distance of 1 km, and d is the distance between transmitter and receiver in kilometers. The signal attenuation rate, or transmission loss rate, is given by γ, which, in practice, takes on values from 0.5 to 3 or more. For VHF and UHF propagation between low antennas over normal ground, $\gamma \approx 2$. Notice that propagation is assumed to be a deterministic function of distance with no provision for location or time variability.[2]

[2]Propagation variability is included in the formal solution (Middleton, 1979a) and will be included in an analytical solution soon (private communication, D. Middleton, 1980).

The distance between the wanted transmitter and the receiver is a random variable, d. It is convenient to use a normalized distance,

$$d_d = d/d_o \quad , \tag{9}$$

where d_o is a reference distance related to the variance of the pdf of d. This pdf is

$$f_d(d_d) = 2\, d_d \exp(-d_d{}^2 - r_{oa}^2)\, I_o(2d_d\, r_{oa}) \quad , \quad d_d \geq 0 \quad . \tag{10}$$

In (10), $r_{oa} = d_a/d_o$, where d_a is the distance between the receiver and the center of operations of the mobile transmitter. Middleton assumes that the mobile transmitter performs a random walk around this center of operations. Naturally, r_{oa} may be zero. $I_0(x)$ is the modified Bessel function of the first kind (Abramowitz and Stegun, 1964).

Figure 1 shows the geometry of the transmitter, T; the receiver, R; and the center of operations, C. Figure 2 illustrates the form of the path length pdf in (10). In Curve A, $d_o = 1$ and $r_{oa} = 0$. Curve B is for $r_{oa} = 0$, but now the reference distance $d_0 = 3$. Comparison of curves A and B shows how d_0 can be used to shape the path length pdf to fit the desired coverage area. Curve C in Figure 2 shows the path length pdf for a center of operations 4 km from the desired receiver.

With these assumptions, the probability that the signal-to-noise ratio exceeds the required signal-to-noise ratio, r, is (Middleton, 1979a)

$$P_a = P(s/i_n \geq r) = 1 - e^{-A} \sum_{m=0}^{\infty} \frac{A^m}{m!} P_m(r) \quad , \tag{11}$$

where

$$P_m(r) = \int_0^{\infty} 2y \exp(-y^2 - r_{oa}^2 - \frac{p_1}{\bar{i}_n r} \frac{g_m}{(d_o y)^{2\gamma}})\, I_o(2r_{oa}y)\, dy \quad . \tag{12}$$

Thus, (11) is an infinite series of definite integrals, which can be summed using computer routines given by Middleton (1979a). (Although this is not an "analytical solution" in this form, Middleton shows that (11) can be reduced to a triple infinite series.)

Recapitulating, the input parameters required to compute P_a with Middleton's (1979a) analytical solution, (11), are

(1) the Class A noise parameters that define the interference environment
  (a) A, the overlap index,

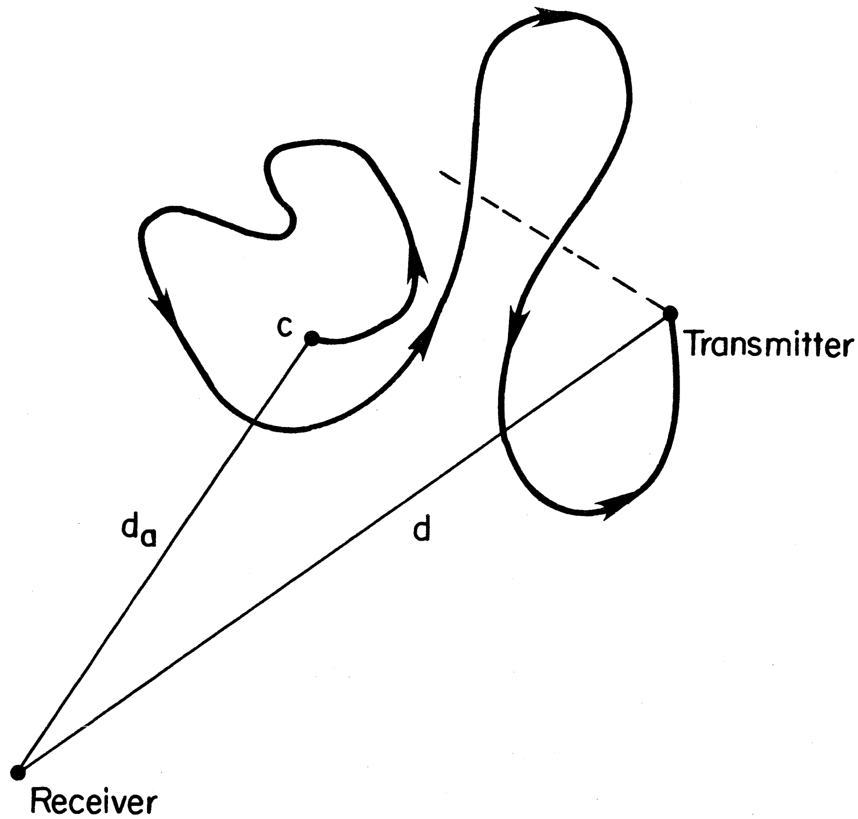

Figure 1. Geometry of wanted path for Middleton's model. The transmitter performs a random walk around its center of operations, C, which is d_a km from the receiver. The random variable d is the distance from the transmitter to the receiver.

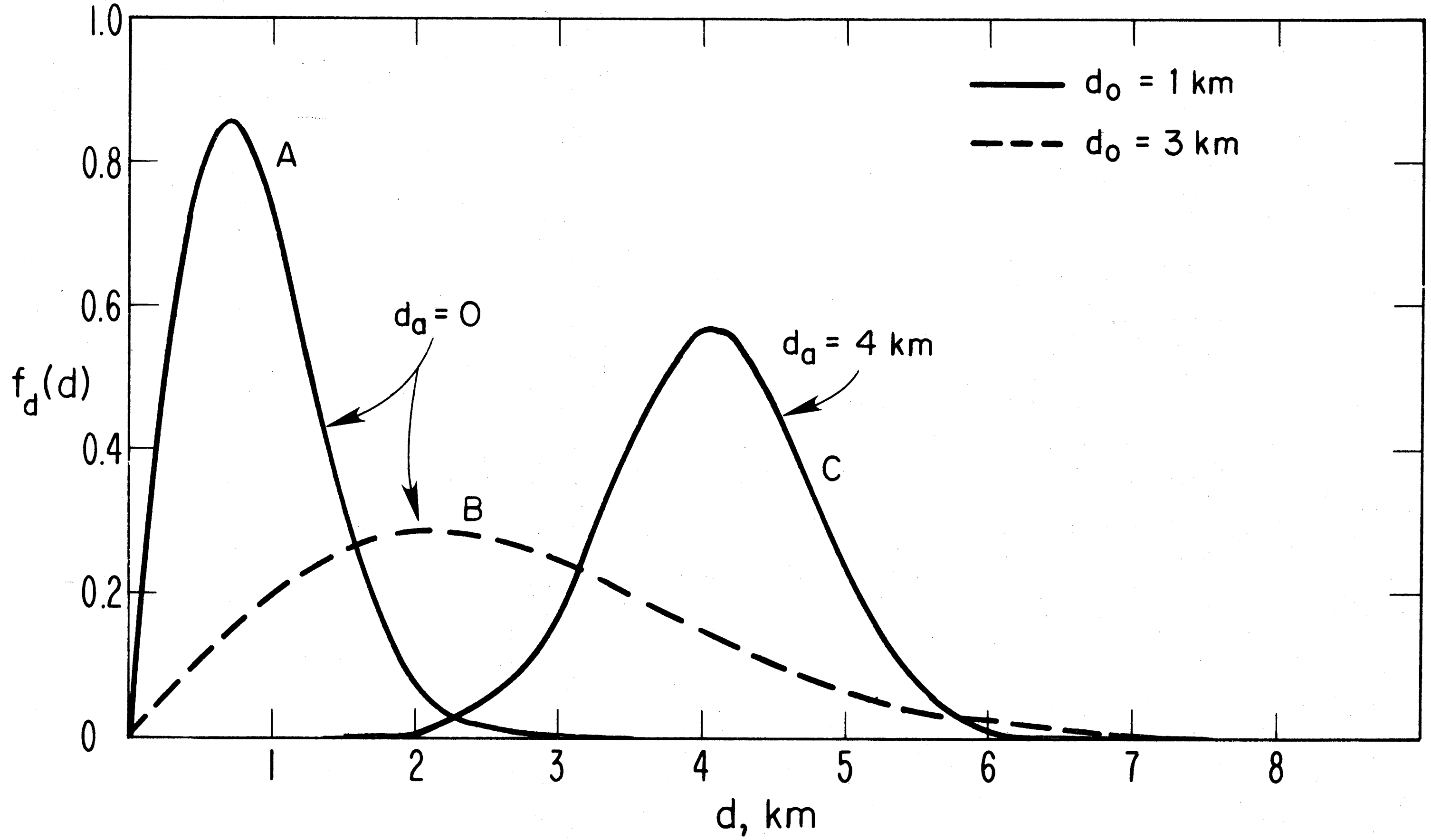

Figure 2. Probability density function of the wanted path length, equation (10). The center of operations of the mobile transmitter is d_a km from the receiver.

(b) $\overline{i}_n$, the mean of the total noise, and

(c) Γ, the ratio of the mean of the gaussian noise to mean of the nongaussian noise;

(2) the path length or geographic parameters for the wanted signal

(a) $r_{oa} = d_a/d_o$, where d_a is the distance between the desired receiver and the center of operations for the transmitter, and

(b) d_o, a reference distance used to fit the pdf to the desired coverage area;

(3) the attenuation rate parameter, γ;

(4) the system parameters

(a) p_1, the transmitter power and

(b) r, the required signal-to-noise ratio. (Actually, the ratio p_1/r is all that is needed.)

2.2 Numerical Model for the Probability of Communications

In many congested spectrum-use environments, there is significant variability in almost every factor influencing communications success. Transmitters may be mobile or their locations may be unknown, either because data are not available or because the systems have not yet been deployed. Antenna heights may depend on availability of favorable locations or on desired coverage of individual systems. Transmitter powers may vary by specification or because of aging and different maintenance procedures. Transmission loss varies with details of the transmission path and often with time because of weather or other geophysical events. Message traffic or channel usage may vary with the requirements of the users. Therefore, any realistic estimate of communications success must be probabilistic. It should be based on consideration of statistical distribution of all significant factors.

The first order description of variation of a random process is provided by the probability density function (pdf). When several random processes interact to produce an effect, that effect is itself a random process with a pdf. If the pdf's of each of the contributing random variables and the nature of their interactions are known, rules such as those shown in Table 2 provide procedures for finding the pdf of the effect.

Analytical closed-form solutions for the desired pdf's are possible for only a few special forms of contributing pdf's because most of the rules involve integration of products. However, numerical integration is an accurate and efficient procedure on electronic computers. Berry (1977) has developed a computer program

Table 2. Elementary Compositions of Random Variables

(Details can be found in Zehna (1970) or other probability theory textbooks.)

Notation: If X is a random variable, $f_X(x)$ is its probability density function (pdf).

$f_X(x) \geq 0$ if $x_1 \leq x \leq x_m$, otherwise $f_X(x) = 0$.

$f_{XY}(x,y)$ is the joint pdf of X and Y. $P(X \leq t) = \int_{x_1}^{t} f_X(x)\, dx$.

$f_{X|Y}(x|y)$ is the conditional pdf of X, given Y.

Transformation: Let g(x) be a monotonic function, and let h(x) be its inverse. If $Y = g(X)$, $f_Y(y) = f_X(h(y))\, |h'(y)|$, where $h'(y)$ is the derivative of $h(y)$.

In particular, if $Y = aX + b$, $f_Y(y) = \frac{1}{a} f_X\left(\frac{y-b}{a}\right)$.

Arithmetic:

If $Z = X + Y$, $A = \max(x_1, z-y_m)$, and $B = \min(x_m, z-y_1)$, then $f_Z(z) = \int_A^B f_{XY}(x, z-x)\, dx$. If X and Y are independent, $f_Z(z) = \int_A^B f_X(x)\, f_Y(z-x)\, dx$.

If $Z = XY$, then $f_z(Z) = \int_A^B \frac{1}{|x|} f_{XY}(x, \frac{z}{x})\, dx$. If X and Y are independent,

$f_Z(z) = \int_A^B \frac{1}{|x|} f_X(x)\, f_Y(\frac{z}{x})\, dx$. If $x_1 > 0$ and $y_1 > 0$, then $A = \max(x_1, z/y_m)$, and $B = \min(x_m, z/y_1)$.

If $Z = X/Y$, then $f_Z(z) = \int_A^B |y|\, f_{XY}(zy, y)\, dy$. If X and Y are independent,

$f_Z(z) = \int_A^B |y|\, f_X(zy),\, f_Y(y)\, dy$. If $x_1 > 0$ and $y_1 > 0$, then $A = \max(y_1, x_1/z)$, and $B = \min(y_m, x_m/z)$.

Conditionals: If X is conditional on Y, $A = \max(x_1, y_1)$, and $B = \min(x_m, y_m)$, then $f_X(x) = \int_A^B f_{X|Y}(x|y)\, f_Y(y)\, dy$.

for calculating the probability of communications that exploits this capability. Because the integration is done numerically, the pdf which best describes the parameter variation can be used as input. The pdf need not even be analytic--a table of measured values can be used as the input pdf. This program has been described in detail by Berry (1977, 1978). Its structure will be briefly outlined in this section.

In Berry's model, all power is expressed in dBW rather than in watts. Although either set of units can be used, most measured data on transmission loss and noise are given in dB, and the pdf's of these measured data are approximately normally distributed in dB.

The formal solution for P_a evaluated by the computer program is

$$P_a = P(S-I \geq R) = \int_R^\infty f_{S-I}(x)\, dx \quad , \tag{13}$$

where

S is the received signal strength in dBW,

I is the received interference in dBW,

R is the required SNR in dBW, and

$f_{S-I}(x)$ is the pdf of $X = S-I$.

Thus (13) is the equivalent of (1) for powers given in dBW. The procedure is to find $f_{S-I}(x)$ numerically.

The overall plan in the program is to compute the pdf of the wanted signal, then compute the pdf's of each "kind" of interference. The powers from the different sources of interference are added together with the ambient noise to get the pdf of the total noise plus interference. This sum is combined with the signal pdf to obtain the probability distribution of the signal-to-noise ratio. The probability of achieving a specified signal-to-noise ratio is then computed. Figure 3 shows the major steps in this calculation.

To compute the distribution of the desired signal at a random receiver, the pdf of wanted-transmitter effective radiated power (erp) must be known. In practice, this has included the effects of antenna gain and antenna height gain, but these factors could be specified separately and used to compute the pdf of erp.

In most terrestrial problems, the transmission loss for a fixed path length is not a constant but varies with time and the specific details of the propagation path. Thus, transmission loss is a conditional random variable with a statistical distribution for each distance. In practice, the mean of the distribution is specified as a function of distance, often with a formula like (8), and the variation around this mean is specified. The conditional pdf of transmission loss can

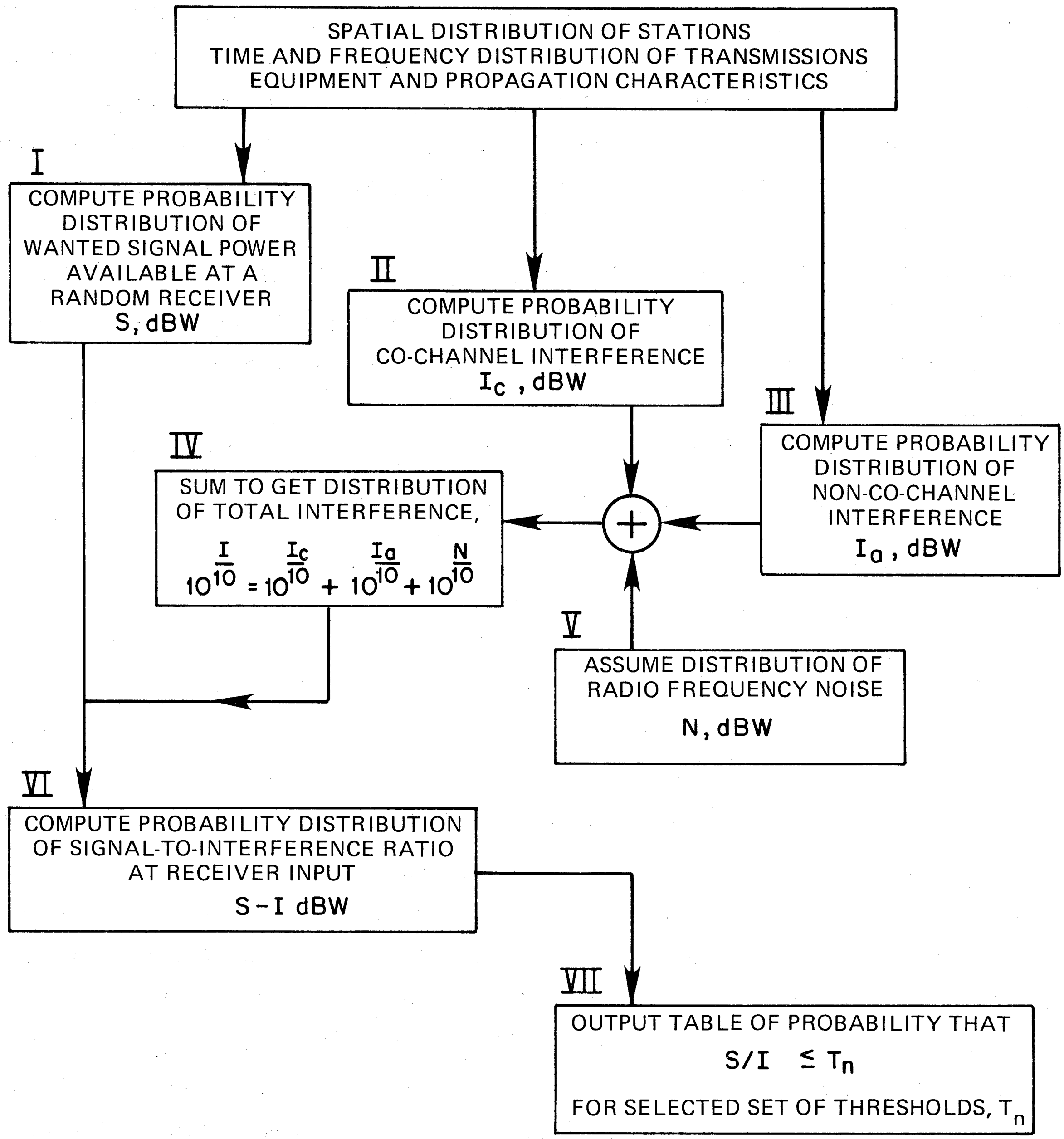

Figure 3. Block diagram of program for computing the probability of interference in a congested environment.

be combined with the pdf of distances (path lengths) using the formula in Table 2 to provide an unconditional pdf of transmission loss. The path length pdf can be specified analytically, as in (10), or numerically. It is often generated off line by using a real or postulated distribution of transmitter locations and tabulating the resulting path lengths.

The signal, S, is the transmission loss subtracted from the transmitter power, so the pdf of the signal available at a randomly selected receiver is computed using the convolution integral given in Table 2.

A similar calculation produces the pdf for each category of interference. Different categories are necessary when different pdf's are necessary to describe any of the input parameters. For example, co-channel and adjacent channel interferers may have different path length pdf's because of circuit discipline (Berry, 1977), and local and skywave interferers have different transmission loss distributions (Berry, 1978). There also may be an additional transformation (in the receiver) for interference such as adjacent channel interference or intermodulation interference (unpublished Technical Memorandum 79-14, "Probability of intermodulation interference in an expanded CB service").

The result of the interference calculation to this point is the pdf of interfering power at a typical receiver given that one randomly selected source in this category is transmitting. It is possible that two, or more, sources are transmitting simultaneously. If so, the two signals have identical pdf's (that is what is meant by being in the same category). The pdf of the sum of the two powers must be found using the rule in Table 2. It is assumed that interfering power from two sources add incoherently. Similarly, the pdf for three or more interferers can be computed; interference from different categories of interferers can be added together; and the sum can be added to the background noise.

To perform these additions, the random variables must be transformed from dBW to watts. If X is a random variable in dBW, with pdf $f_X(x)$, let $Y = 10^{X/10}$, so that Y is the corresponding value in watts. The pdf of Y (using the transformation rule in Table 2) is

$$f_Y(y) = \frac{10 \log_{10} e}{y} f_X(10 \log_{10} y) \quad . \tag{14}$$

After adding the various contributions to the noise, the pdf of the sum, Z, can be transformed back to dBW. Let $V = 10 \log_{10} Z$. Then the pdf of V is

$$f_V(v) = D\, 10^{v/10} f_Z(10^{v/10}) \quad , \tag{15}$$

where $D = (\log_e 10)/10$. This is the pdf of interference, given a specific number of interferers.

The unconditional pdf of interference is computed as follows. In general, the probability that the interference is below a given value t is

$$P(I_N \leq t) = P([I_N \leq t \text{ and } 0 \text{ on}] \text{ or } [I_N \leq t \text{ and } 1 \text{ on}] \text{ or} \ldots$$

$$\ldots \text{or } [I_N \leq t \text{ and } M \text{ on}]) \quad , \tag{16}$$

where M is the total number of potential interferers. In (16), the phrase "m on" means "exactly m transmitters are transmitting."

Because the events in square brackets are mutually exclusive,

$$P(I_N \leq t) = \sum_{m=0}^{M} P(I_N \leq t \text{ and } m \text{ on}) \quad , \tag{17}$$

$$= \sum_{m=0}^{M} P(I_N \leq t | m \text{ on}) \, P(m \text{ on}) \quad . \tag{18}$$

The probability, P(m on), depends on the average traffic intensity of each link and on the operating practices in the service. In a disciplined land-mobile service, a user will usually wait until the channel seems to be clear before transmitting. This means that P(m) is very small for $m \geq 2$. P(m) for $m \geq 2$ would be zero if every operator had perfect information about others' use of the channel. However, there is a possibility that an operator will not hear another transmitter even though it is transmitting and will then emit some interference. Thus, computing P(m on) is quite difficult in a disciplined service.

In a completely undisciplined service, it is easier to model P(m on) because transmission times are random and independent. Under quite general conditions then, the number of transmitters on at any instant of time is poisson distributed. Examples of the mathematical assumptions that are required are given by Zehna (1970, p. 135) and Kleinrock (1975, p. 60). In the present application, the assumptions mean that transmitter turn-ons are independent (no courtesy) and that the number of potential interferers is large compared to the average number transmitting at any given instant. These are the assumptions made by Middleton (1972, 1974, 1976, 1979a). Under these conditions (Kleinrock, 1975),

$$P(m \text{ on}) = \frac{U^m}{m!} e^{-U} \quad . \tag{19}$$

In (19),

$$U = \nu \overline{T} \quad , \tag{20}$$

where ν is the average number of transmitter turn-ons per second, and $\overline{T}$ is the average length of a transmission.

Substituting (19) into (18) yields

$$P(I_N \leq t) = \sum_{m=0}^{M} e^{-U} \frac{U^m}{m!} P(I_N \leq t|m \text{ on}) \quad . \tag{21}$$

$P(I_N \leq t)$ is of course the probability distribution of I_N. Taking the derivative with respect to I_N results in the pdf:

$$f_I(I_N) = \sum_{m=0}^{M} e^{-U} \frac{U^m}{m!} f_{I|m}(I_N|m) \quad . \tag{22}$$

When 0 transmitters are on, the noise consists entirely of the background or ambient noise. The pdf of this noise, $f_{I|0}$, must be supplied as input. When one or more transmitters are on, there is noise plus interference, and the pdf is computed using the process described above culminated by equation (15). Thus, equation (22) is computable and is the pdf of interference-plus-noise for given traffic intensity, U, and number of potential interfering transmitters, M. Theoretically, M can be infinite. Table 3 shows values of the coefficients of the pdf in (22) for various traffic intensities, U. In land mobile radio channels, U is usually less than one.

The pdf of the signal-to-interference ratio can now be computed for the random variable S-I using the convolution integral shown in Table 2. The probability of noninterference is the probability that S-I exceeds the required value R and is given by (13).

Recapitulating, the input allowed in the numerical model is

(1) for the wanted signal,
  (a) the pdf of radiated power of the wanted transmitters,
  (b) the conditional pdf of transmission loss, given wanted path length,
  (c) the pdf of wanted path length;
(2) for each category of interferer,
  (a) the pdf of radiated power,
  (b) the conditional pdf of transmission loss, given interfering path length,

Table 3. Probability that i (Out of Possible m) Transmitters Assigned to a Channel are Transmitting

(U is the total channel utilization, U=(Jℓ/60) m, where J is number of transmissions per hour by a transmitter and ℓ is average transmission length in minutes.)

i	U, Channel Utilization							
	0.25	0.5	1	2	3	4	5	10
0	.7788	.6065	.3679	.1353	.0498	.0183	.0067	$4.54(10^{-5})$
1	.1947	.3033	.3679	.2707	.1494	.0733	.0337	.0005
2	.0243	.0758	.1839	.2707	.2240	.1465	.0842	.0023
3	.0020	.0126	.0613	.1804	.2240	.1954	.1404	.0076
4	.0001	.0016	.0153	.0902	.1680	.1954	.1755	.0189
5	.0000	.0002	.0031	.0361	.1008	.1563	.1755	.0378
6		.0000	.0005	.0120	.0504	.1042	.1462	.0631
7			.0001	.0034	.0216	.0595	.1044	.0901
8			.0000	.0009	.0081	.0298	.0653	.1126
9				.0002	.0027	.0132	.0363	.1251
10				.0000	.0008	.0053	.0181	.1251
11					.0002	.0019	.0082	.1137
12					.0001	.0006	.0034	.0948
13					.0000	.0002	.0013	.0729
14						.0001	.0005	.0521
15						.0000	.0002	.0347
16							.0000	.0217
17								.0128
18								.0071
19								.0037
20								.0019
21								.0009
22								.0004
23								.0002
24								.0001
25								.0000

(c) the pdf of path lengths,

(d) the traffic intensity, U, of interferers in this category,

(e) the receiver transfer function, if any;

(3) the pdf of ambient noise; and

(4) the required signal-to-noise ratio.

Any pdf may be given analytically, in which case the defining parameters (e.g., mean, variance) must be given, or the pdf may be defined by an input table of measured values.

2.2.1 Specialization of the Numerical Model for Comparisons

To compare output from the numerical model with calculations made using equation (11), two modifications are necessary. First, the calculation of the pdf of noise plus interference from input descriptors of the power, locations, etc. of the interferers, (22), is replaced by a subroutine to calculate the pdf of Class A noise, (2).

Second, the calculation of wanted signal is simplified because the analytical solution, (11), assumes that there is no variability in radiated power and the transmission loss is a deterministic (rather than probabilistic) function of path length. Converting the signal in watts (equation (8)) to dBW,

$$S = 10 \log_{10} p_1 - 20 \gamma \log_{10} d \quad , \qquad \text{dBW} \quad . \tag{23}$$

But d is a random variable, so S is a random variable whose pdf is found using the transformation rule in Table 2. It is

$$f_S(S) = D\left(10^{(B-S)/20\gamma}\right) f_d\left(10^{(B-S)/20\gamma}\right) \quad , \tag{24}$$

where $B = 10 \log_{10} p_1$ and $f_d(x)$ is given by (10).

2.2.2 The pdf of Class A Noise Given in dBW

The numerical model treats power in dBW, so the pdf of Class A noise, equation (2), must be transformed as shown in Table 2. This can be done mechanically by the computer, but it is instructive to perform the transformation analytically. Let $I_N = 10 \log_{10} i_n$. Then the pdf of I_N is

$$f_I(I_N) = e^{-A} \sum_{m=0}^{\infty} \frac{A^m}{m!} \frac{g_m}{\bar{i}_n} D \exp\left(DI_N - \frac{g_m}{\bar{i}_n} e^{DI_N}\right) \quad , \tag{25}$$

where $\bar{i}_n$ is the mean of total noise power, i_n, in watts. (Remember that $D = (\log_e 10)/10$.)

An approximation appropriate for land-mobile radio channels results from assuming that the mean of the background (gaussian) noise is much less than the mean of the co-channel interference (nongaussian noise) and that the traffic intensity (channel occupancy) is less than one. That is, assume that

$$\Gamma \ll A < 1 \quad . \tag{26}$$

In (3), then,

$$g_0 \approx 1/\Gamma \text{ and}$$

$$g_m \approx A/m \text{ if } m \neq 0 \quad . \tag{27}$$

Equation (6) shows that

$$\bar{i}_n \approx \bar{i}_c \tag{28}$$

when $\Gamma \ll 1$, so that

$$g_0/\bar{i}_n \approx \left(\bar{i}_n \frac{\bar{n}}{\bar{i}_n}\right)^{-1} = 1/\bar{n} \quad . \tag{29}$$

To the approximation in (26), the first term of the series in (25) is a function of only the mean of the gaussian noise, $\bar{n}$, and thus, it is the pdf of the background noise in dBW.

Figure 4 shows a plot of $f_I(I_N)$ for $A = 0.35$, $\Gamma = 0.0005$, and $\bar{i}_c = 10^{-10}$ watts, or $\bar{I}_c = -100$ dBW. These values are "in the ball park" for low-band VHF land-mobile radio. The gaussian and nongaussian noise are clearly separated in Figure 4. The bump on the left is the probability density of the background noise, and bump on the right is the probability density of the interference--the nongaussian noise. Analysis of the details of these bumps presents a clear physical picture.

First, the maximum of each term in (25) can be found as a function of I_N by equating the derivative to 0. The result is

$$I_N(m) = 10 \log_{10}(\bar{i}_n/g_m) \quad , \tag{30}$$

where $I_N(m)$ denotes the value of I_N at which the m^{th} term is a maximum. For m=0 and the approximation in (27), the maximum is at

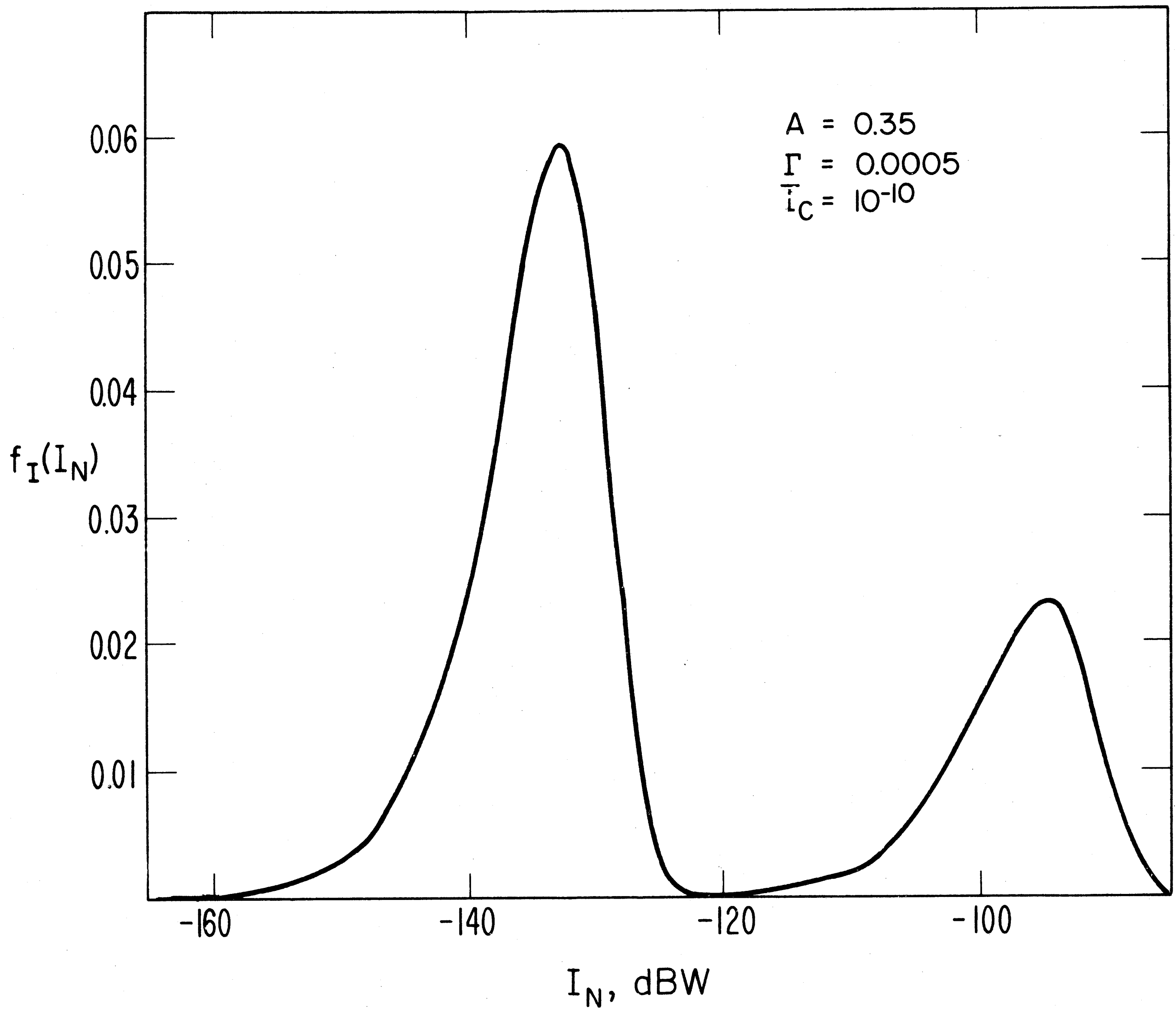

Figure 4. Probability function for intensity of Class A noise in dBW for the parameters shown on the figure. The bump on the left is the probability density of the gaussian noise; the bump on the right is the nongaussian noise or interference.

$$I_N(0) \approx 10 \log_{10} \bar{i}_n \Gamma = \bar{I}_N + 10 \log_{10} \Gamma \quad , \tag{31}$$

which for the parameters of Figure 4 is -133 dBW.

For $m > 0$ and $A << \Gamma < 1$,

$$I_N(m) \approx 10 \log_{10} m \, \bar{i}_C / A \quad . \tag{32}$$

Now $\bar{i}_C$ is the mean interference power averaged over time, and, roughly speaking, A is the fraction of time the nongaussian interference is present[3] (see Table 1). So $\bar{i}_C/A$ is the mean nongaussian interference power averaged over only the time that it is present. The maximum of the m=1 term is at this value in dBW. Notice that the maximum of the nongaussian interference bump in Figure 4 is near -95 dBW, the value $I_N(1)$. This suggests that the higher order terms have only a second order effect on the value of the interference in this case.

Figure 5 shows plots of f_I for two other sets of values of the parameters of Class A noise. As expected from the above analysis, the distance between the two maxima is determined by the ratio of the two means, Γ, in dBW. If $\Gamma > 0.1$, the two bumps merge. The absolute location is determined by $\bar{I}_N$. The ratio of the area of the interference bump to the area of the noise bump is $e^A - 1$. If $A << 1$, this ratio is roughly equal to A--the fraction of time the interference is on.

2.3 Numerical Comparisons of the Two Models

Middleton (1979a) shows plots of the probability of noninterference, $P_a = P(s/i \geq r)$, for various values of the input parameters. P_a was calculated for identical input with the computer program described in Section 2.2 using the specializations in 2.2.1. Specifically,

(1) the pdf of wanted path lengths was given by (10),

(2) the propagation law was deterministic and was given by (8), and

(3) interference had a Class A pdf, given by (25).

Examination of the computer program in Appendix A-IV of Middleton's report (1979a) shows that he converts $y = \hat{x}_0/\bar{x}_0$ to dB by taking 20 times the $\log_{10}$ of it. The ratio itself is dimensionless, but the elementary factors in the ratio are in watts. Therefore, it seems more conventional to define dB = 10 $\log_{10}$. This latter definition is used in this report, and Middleton's results have been replotted to

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[3]More accurately, the fraction of time that interference is present is $1 - P(0 \text{ on}) = 1 - e^{-A}$ (see (19)). But when $A << 1$, $e^{-A} \approx 1 - A$, so $1 - P(0 \text{ on}) \approx A$.

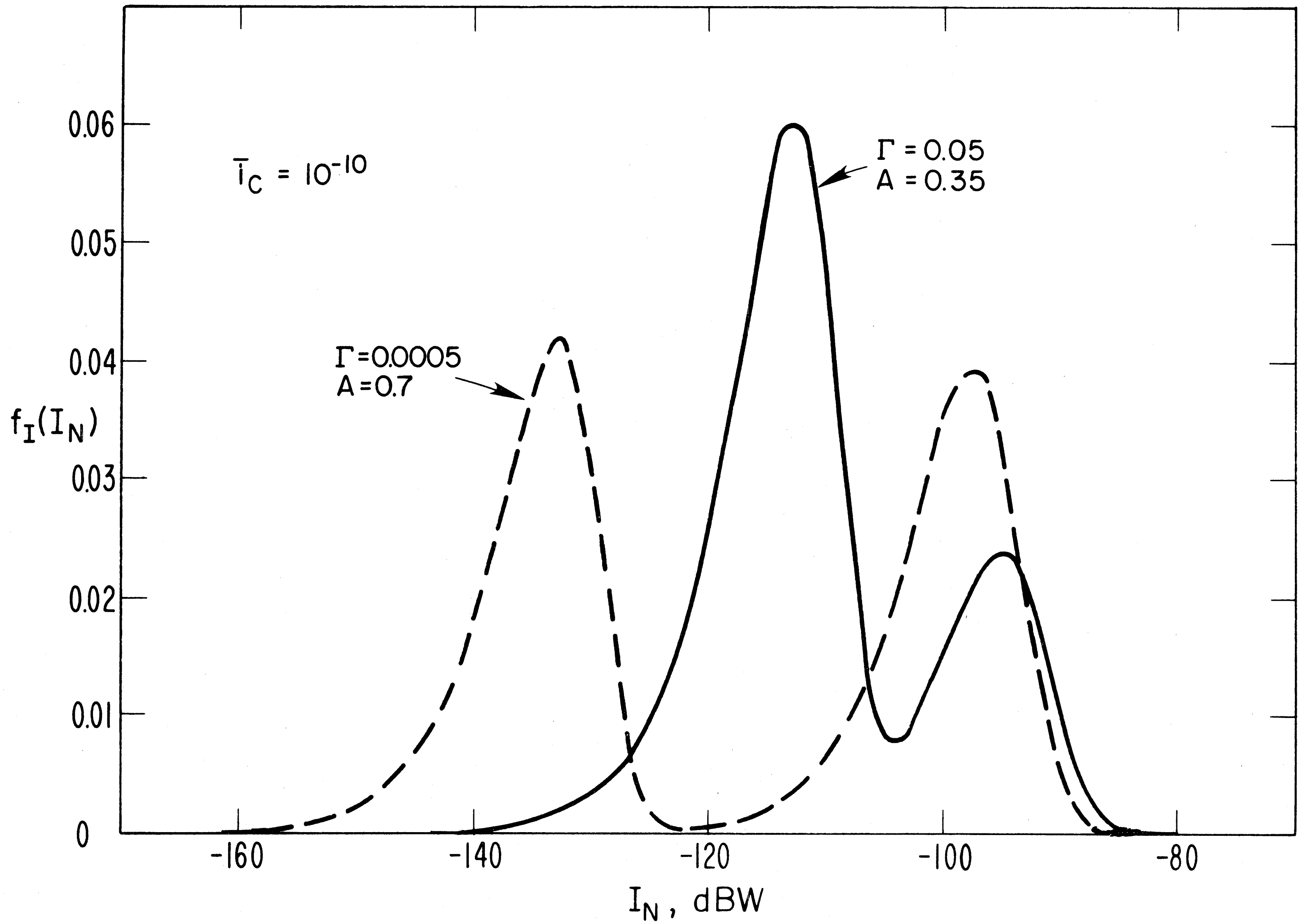

Figure 5. Probability density function for Class A noise in dBW for the parameters shown on the figure. Comparison with Figure 4 shows that increasing Γ moves the bumps closer together; increasing A changes the relative sizes of the bumps.

this scale. Middleton plots P_a over a limited range of the independent variable. This range is extended in this report by using the program listed in Middleton's report to compute additional values.

Figure 6 shows a comparison of the output of the two methods for a propagation parameter γ = 2 and for two values of the noncentral displacement, r_{0a}. The two methods give essentially identical results, except for very small P_a.

The shape of the curves can be understood by referring to Figure 4, which is the pdf of Class A noise for the same case as in Figure 6. Remember that the bump on the left in Figure 4 is the continuous background noise, while the bump on the right is the intermittent interference present about 30 percent of the time. Assume that the observed transmitter 1 km away has controllable power, that you are measuring the signal-to-noise ratio at the receiver, and that the required signal-to-noise ratio is 20 dB. Then in Figure 6, y = received signal +80 dB.

Begin with a low radiated power so that the received signal is less than -140 dBW. Then interference occurs unless the noise is less than -160 dBW. Figure 4 shows that this is very improbable, so $P_a \approx 0$. Now increase the power 10 dB so the received signal is -130 dBW. Then the required S-I is achieved when the noise is -150 dBW. The probability of this occurrence is the area under the curve in Figure 4 to the left of -150 dBW, a small but nonzero probability. As the power is increased more and more, moving right on Figure 4, the area under the curve left of the threshold increases, and P_a increases until the received signal power is -105 dBW. At this point, S-I = 20 dB if the noise is -125 dBW. As the power is increased 10 or 15 dB above this level, P_a remains almost constant, because there is virtually no area under the curve in this region. The received signal is powerful enough that it exceeds the background noise nearly all the time. However, P_a is about 0.7 rather than one, because the signal is too weak to compete with the interference whenever the interference is present.

As the power is increased still further, so that the signal is 20 dB greater than the noise at -110 dBW, P_a begins to increase again. The signal is now strong enough to beat the interference part of the time. P_a continues to increase with increasing transmitter power, until the received signal is about -65 dBW. At this power, S-I is greater than 20 dBW nearly all the time, and $P_a \approx 1$. Even when the interference is on, the signal is more than 20 dB stronger.

So when the mean of the gaussian noise is much less than the mean of the intermittent interference, P_a will be small when y is small; increase to a plateau near $1-e^{-A}$ as y increases; remain nearly constant for a while; and then increase again to a value near one. This is exactly the behavior of the curves in Figure 6.

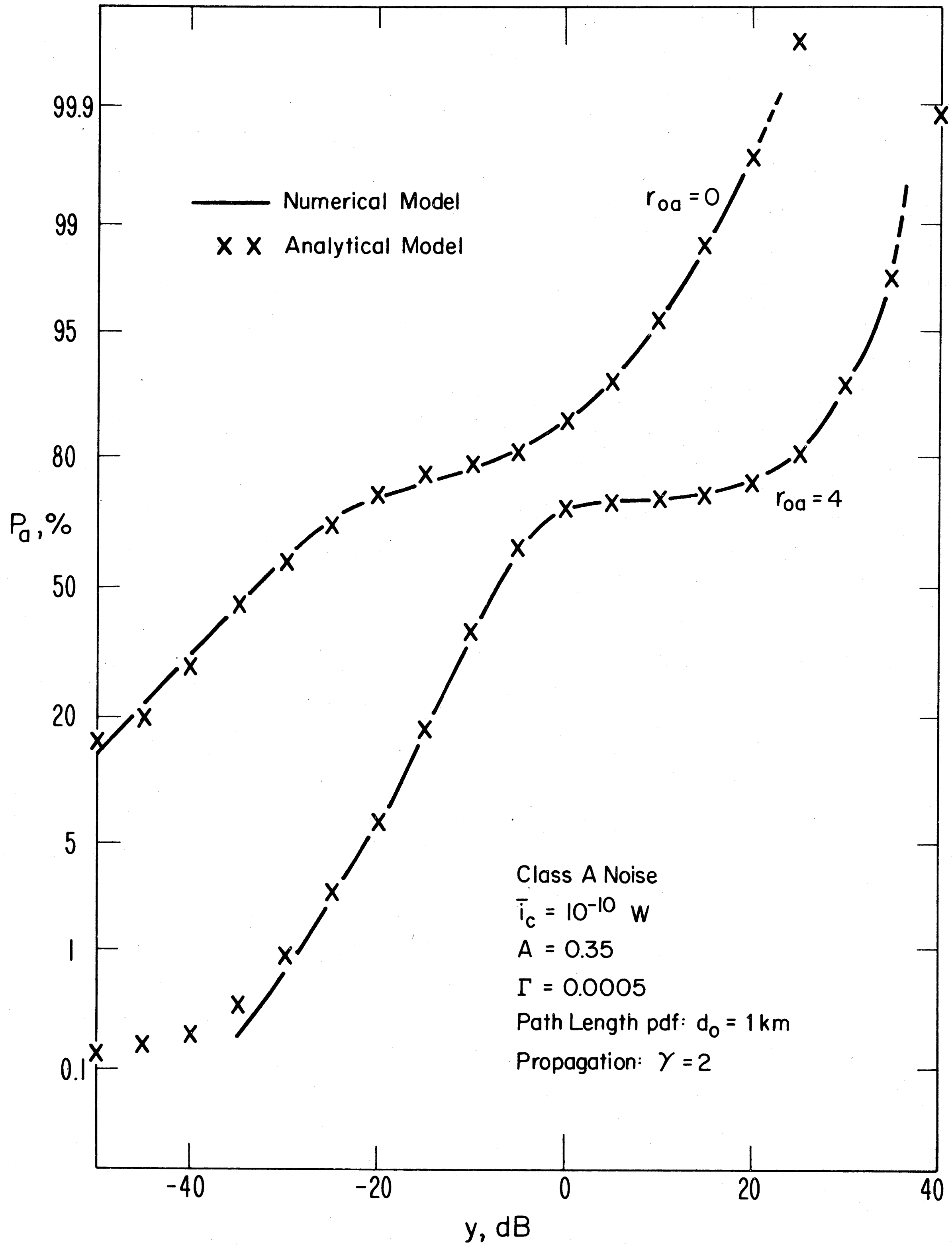

Figure 6. Probability, P_a, that the signal-to-noise ratio exceeds the required threshold as a function of Middleton's parameter $y = \hat{x}_0/\bar{x}_0$, dB. Notice the plateau in P_a near $P_a = e^{-A}$. The propagation factor γ=2.

In the thought experiment above, the distance between the receiver and transmitter is fixed. It is a random variable for the calculations that led to Figure 6 so there is a range of signal levels rather than a fixed value, to compare to the noise. This results in some smoothing of the plateau effect because, when the mean of the received signal is between the bumps in Figure 4, the tails of the signal distribution extend into both bumps. To illustrate, Figure 7 shows the distribution of the received signal for both cases shown in Figure 6. The distribution of the received signal for $r_{oa} = 0$ is over 30 dB wide, easily bridging the 15 dB interval between the bumps in Figure 4. Thus, the plateau in the curve for $r_{oa} = 0$ in Figure 6 is not very pronounced.

The signal distribution of the received signal for a center of operations displaced 4 km from the receiver ($r_{oa} = 4$) is much narrower. This is because the ratio of farthest-to nearest-probable transmitter location is not so large for this situation (see Fig. 2). The effective range of the signal distribution for $r_{oa} = 4$ is about 15 dB (Fig. 7), which fits neatly in the gap between the bumps in Figure 4. Therefore, the plateau in P_a is quite noticeable in the curve for $r_{oa} = 4$ in Figure 6.

For free-space propagation ($\gamma = 1$), received signal strength does not vary so much with path length, so the pdf of the signal is narrower in Figure 7. As a result, the plateau in P_a is more pronounced in Figure 8, which compares the analytical and numerical methods for $\gamma = 1$.

Without comparing curves for all of the parameter combinations used by Middleton, it is clear that the two methods produce the same probability of communications for the same input parameters.

3. EFFECTS OF DIFFERENT WANTED SIGNAL INPUT ASSUMPTIONS ON P_a

Comparison of the summaries at the end of Sections 2.1 and 2.2 shows that the available computer program allows more freedom in the specification of input parameters than does Middleton's analytical solution, equation (11). Middleton's formal solution, (1), is less specialized but would require a (as yet unwritten) computer program for evaluation. One set of input parameters determines the pdf of the wanted signal. Differences in the probability of communications, P_a, resulting from greater flexibility in this set of input parameters are explored in this section. The even greater differences in ways of specifying the interference and noise environment are examined in Section 4.

3.1 The Effects of Variability in Transmission Loss

The analytical solution (11) assumes that transmission loss is a deterministic function of path length given by equation (8) (see footnote 2, page 7). However, if

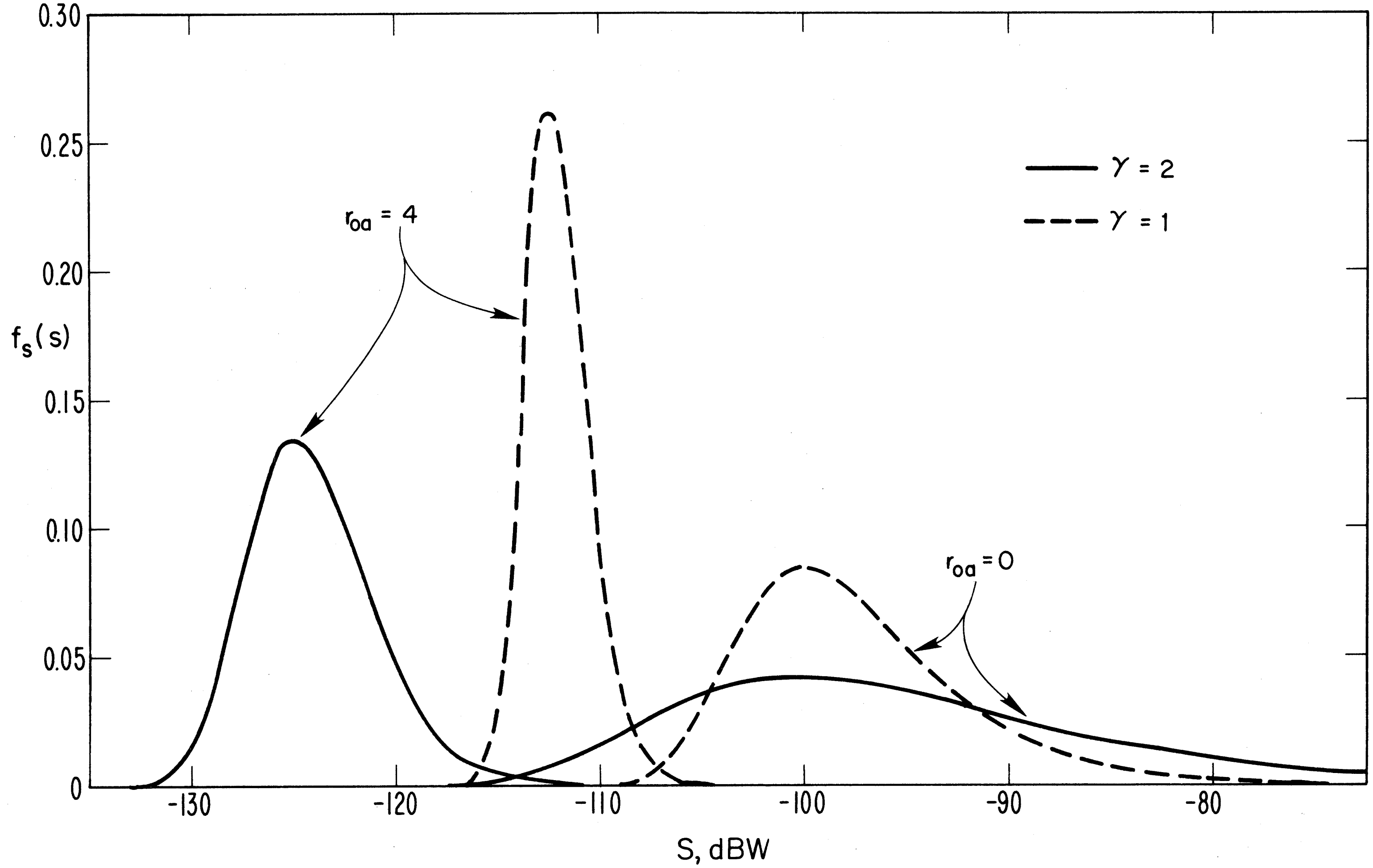

Figure 7. Probability density function of received signal power S for different values of the propagation parameter and for different path length pdfs. Notice that the distributions are narrower for γ=1 than for γ=2 and are narrower for d_a=4 than for d_a=0.

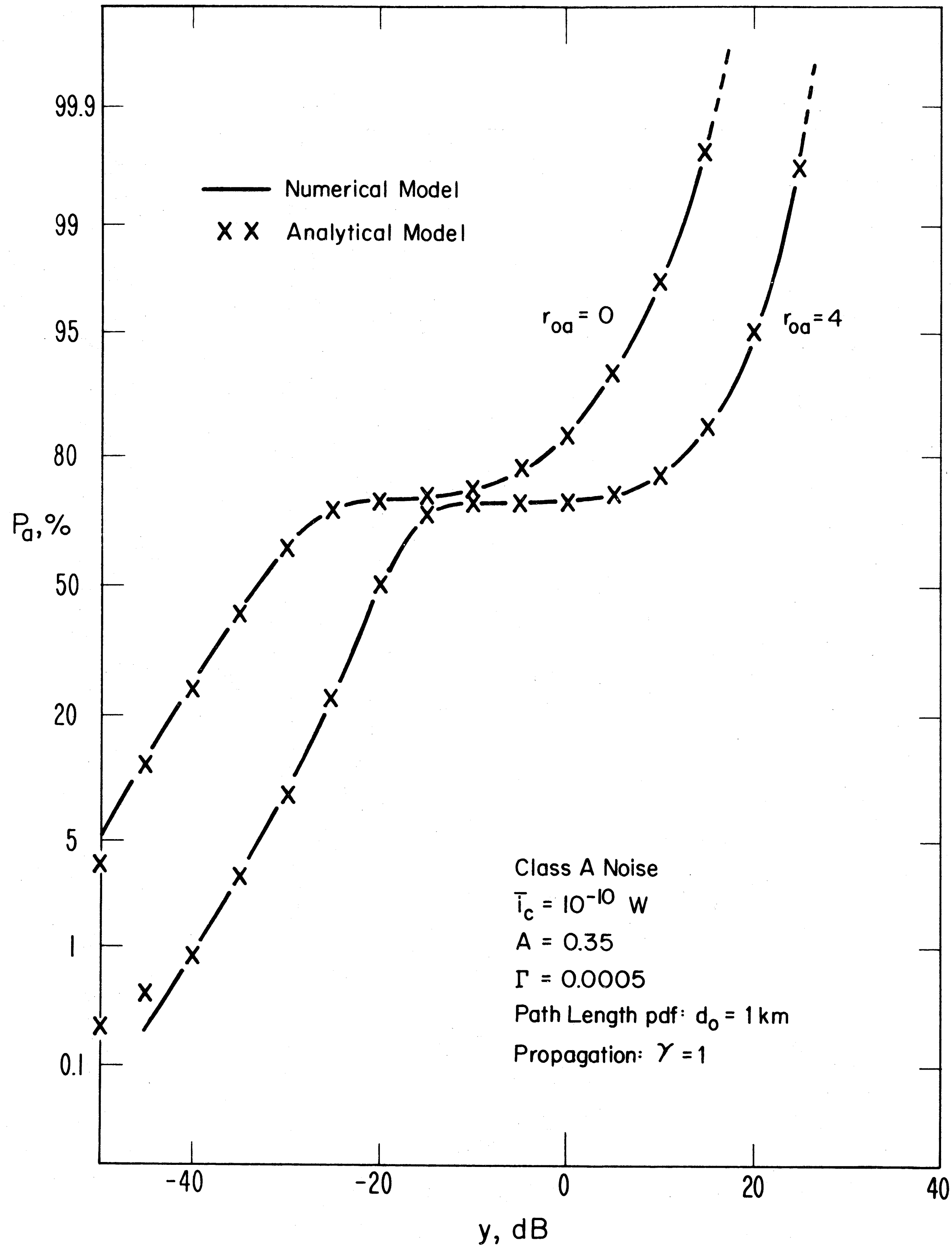

Figure 8. Probability, P_a, that the signal-to-noise ratio exceeds the required threshold as a function of Middleton's parameter, $y = \hat{x}_o/\bar{x}_o$, dB. The propagation parameter γ=1.

received signal power is recorded on a fixed path, it varies with time as atmospheric conditions change. Signals over different paths of the same lengths vary even more because of differences in terrain and man-made structures (see, for example, Barsis, 1971; and Okumura, et al., 1968). This means that received signal power is a probabilistic function of distance with a mean value that may vary with distance like (8) and an approximately log-normal distribution at a fixed distance.

Figure 9 shows how realistic variability in transmission loss affects the probability of communications. The mean value of received power is given by (8) with $\gamma = 1$. The received power in dBW at a fixed distance is assumed to have a normal distribution with a standard deviation of 8 dB, a value appropriate for VHF propagation in nonurban areas (Longley, 1976). The dashed lines in Figure 9 are copied from Figure 8 to facilitate comparisons.

The most obvious effect of the variability in transmission loss is to smooth out the plateau in P_a. The variability of transmission loss for any given path length obviously broadens the distribution of received signal strength so that it bridges the gap between the two bumps in Figure 4. This broadening is shown in Figure 10 where the pdfs of received signal with transmission loss variability are compared to the pdfs of received signal power without variability.

The important practical effect of transmission loss variability shown in Figure 9 is the lowering of the probability of communications for high transmitter power. For example, the probability of communications when $r_{oa} = 4$ and $y = 25$ dB is 99.7 percent without transmission loss variability, but is only 95.7 percent with variability. In other words, inclusion of transmission loss variability in the calculation increases the probability of interference in this region by more than an order of magnitude--from 0.3 percent to 4.3 percent. The messages lost were transmitted over paths with transmission loss much greater than average.

Figure 11 compares P_a with and without transmission loss variability for the more realistic value of $\gamma = 2$. The plateau in P_a is not so prominent for this case where there is no variability, so the inclusion of variability does not change the curves so dramatically. There is still a decrease in P_a for reliabilities higher than 90 percent. However, the curves are close enough together to justify ignoring transmission loss variability (and using the faster Middleton model) for estimates of P_a when reliabilities less than 95 percent are acceptable.

3.2 Effects of Variability in Effective Radiated Power

Often the probability of interference for a category of wanted transmitters is of more interest than the probability of interference for a single transmitter. In this case, the effective radiated power will vary from transmitter to transmitter

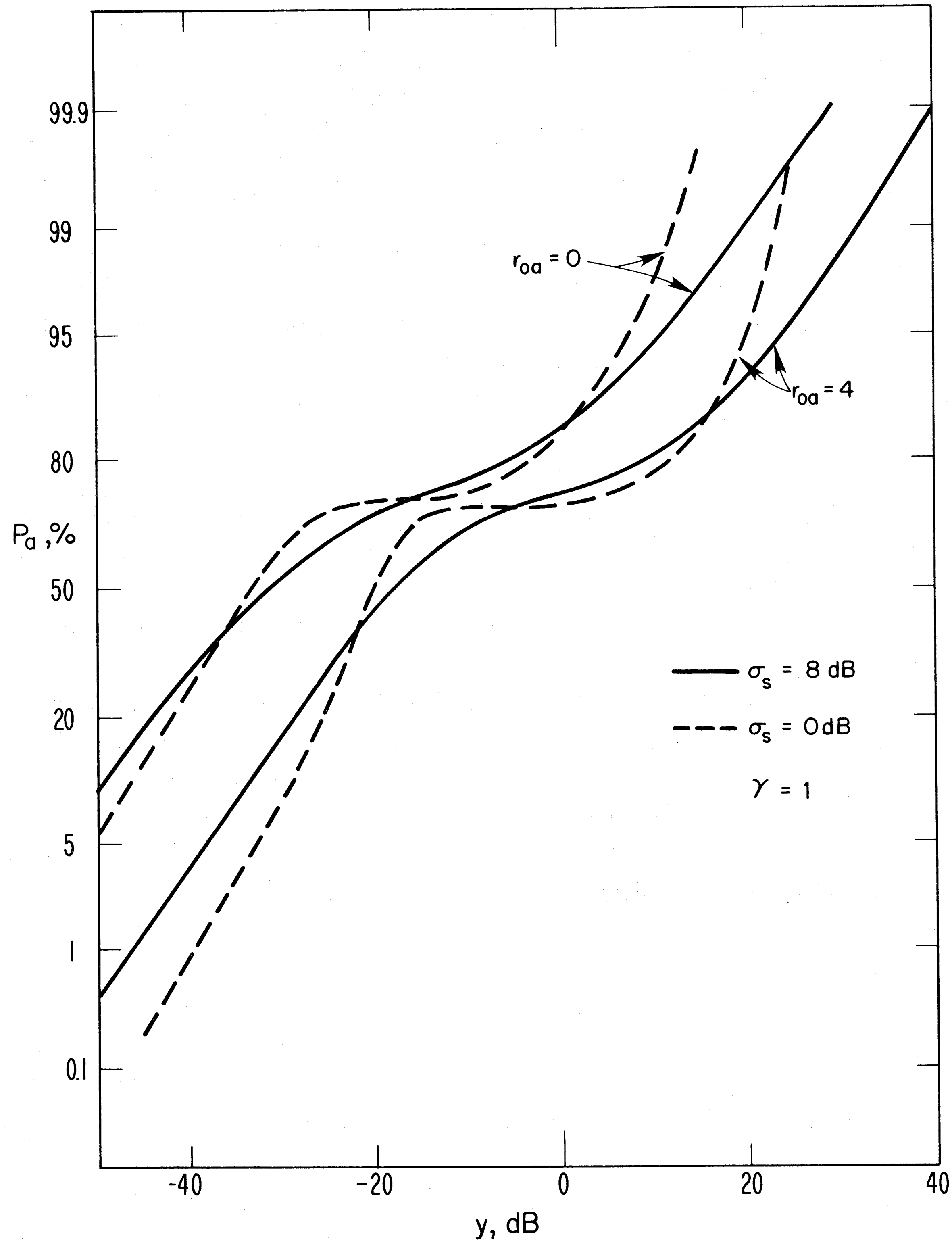

Figure 9. Probability of satisfactory reception, P_a, with and without (dashed line) location variability of transmission loss. Variability of transmission loss smooths out the plateau in P_a. The propagation factor γ = 1.

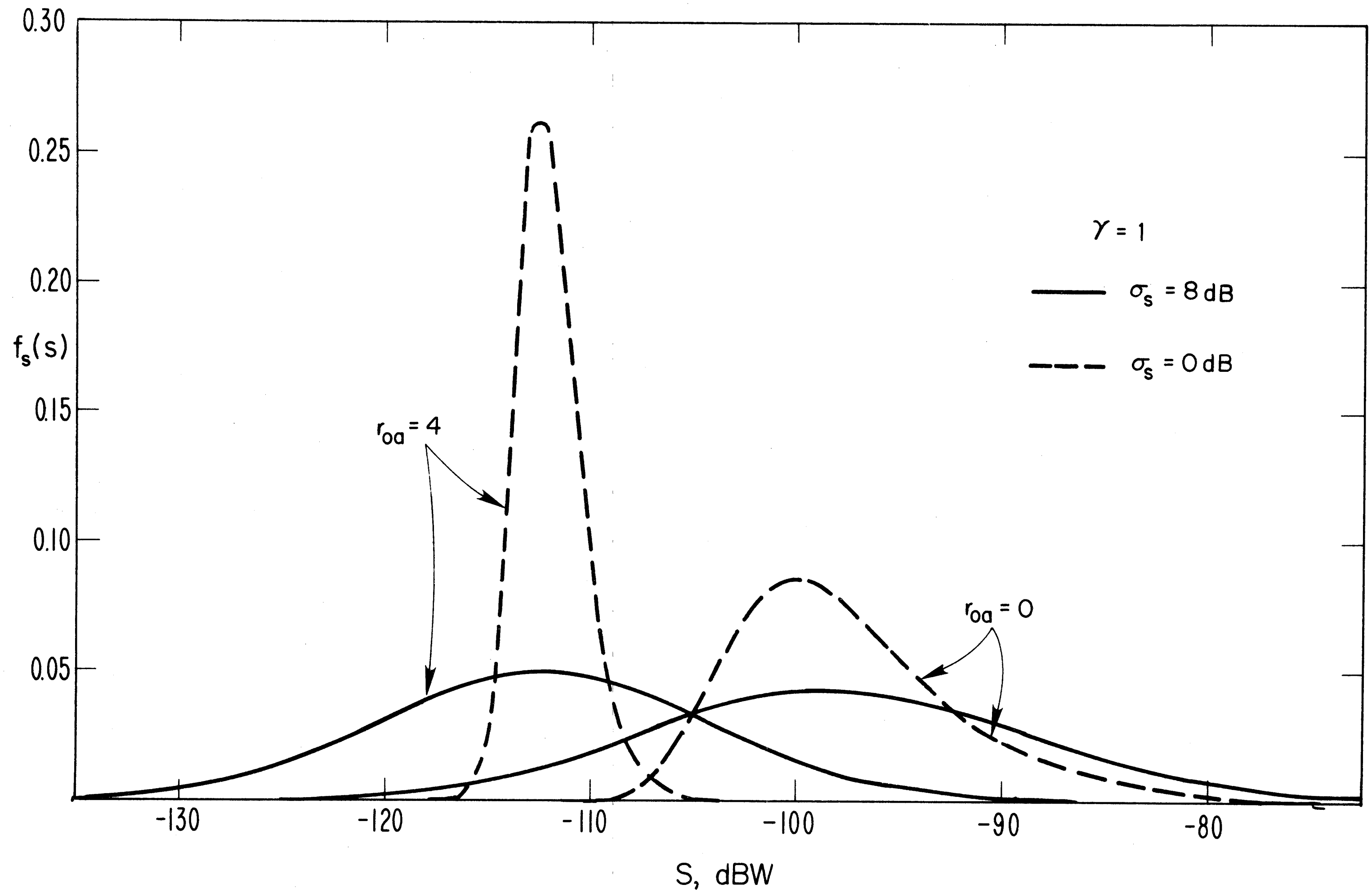

Figure 10. Probability density function of received signal power with and without variability of transmission loss. Nonzero variance broadens the distribution.

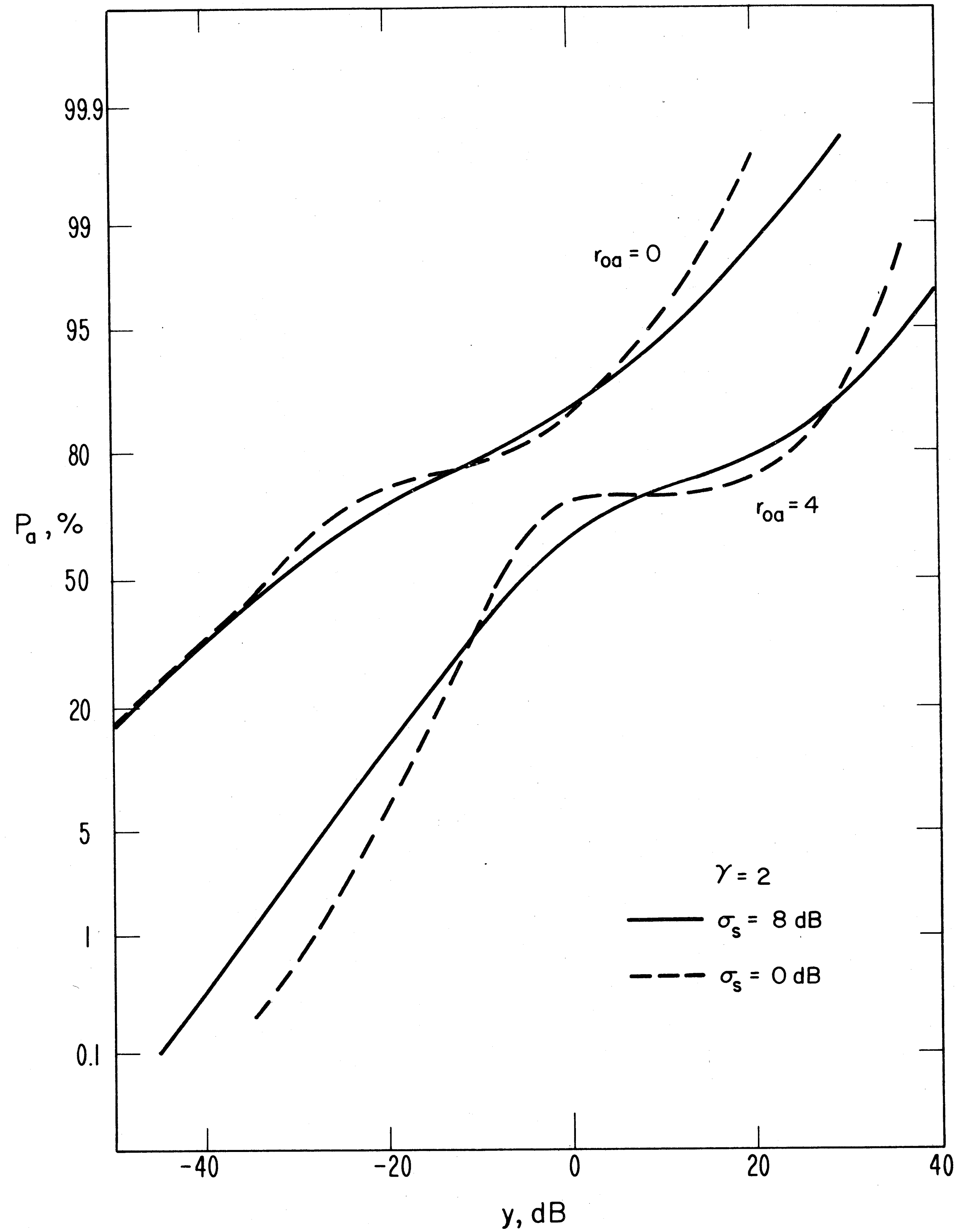

Figure 11. Probability of satisfactory reception, P_a, with and without (dashed line) transmission loss variability for propagation parameter γ = 2.

because of different antenna types, different antenna heights above terrain, and different designs or maintenance of transmitters. The numerical model described in Section 2.2 accepts a distribution of transmitter power as input, but the analytical solution described in Section 2.1 does not.

Calculations are not necessary to estimate the effect of this difference because variability of radiated power simply increases the variability of received signal power just as variability of transmission loss does. A broader distribution of received signals would further smooth out the plateau in P_a and decrease P_a for large values of y.

3.3 Effect of Different Desired Path Length pdfs

To use the analytical solution (11), the probability density function of wanted path lengths must be given by (10). There are two adjustable parameters in (10), so many actual distributions of wanted path lengths can probably be approximated by it. The numerical model accepts any pdf of path lengths as input so it can be used to test the effect of different pdfs on the probability of communications.

The simplest model assumes that wanted transmitters are poisson-distributed in a circle around the wanted receiver. This means that the probability that any small area contains the wanted transmitter is proportional only to the size of the area. Then the pdf of path lengths is

$$f_d(d) = \frac{2}{d_m^2} d \quad , \tag{33}$$

where d_m is the radius of the circle.

Figure 12 compares P_a for the parameters given in Figure 6 with P_a for the same parameters except that (10) is replaced by (33) with d_m=1. (This corresponds to d_o = 1 in (10).) The probability of communications is higher for the calculation using (33) because no paths are longer than 1 km. There are some paths longer than 1 km when (11) is used. The signal over these longer paths is less likely to be strong enough to overcome the noise.

The only point of this comparison is that the probability of communications depends on the form of the pdf of path lengths. The available analytical solution, (11), is restricted to one form of path length pdf--that given by (10). The computer program can use whatever path length pdf best fits the operational situation.

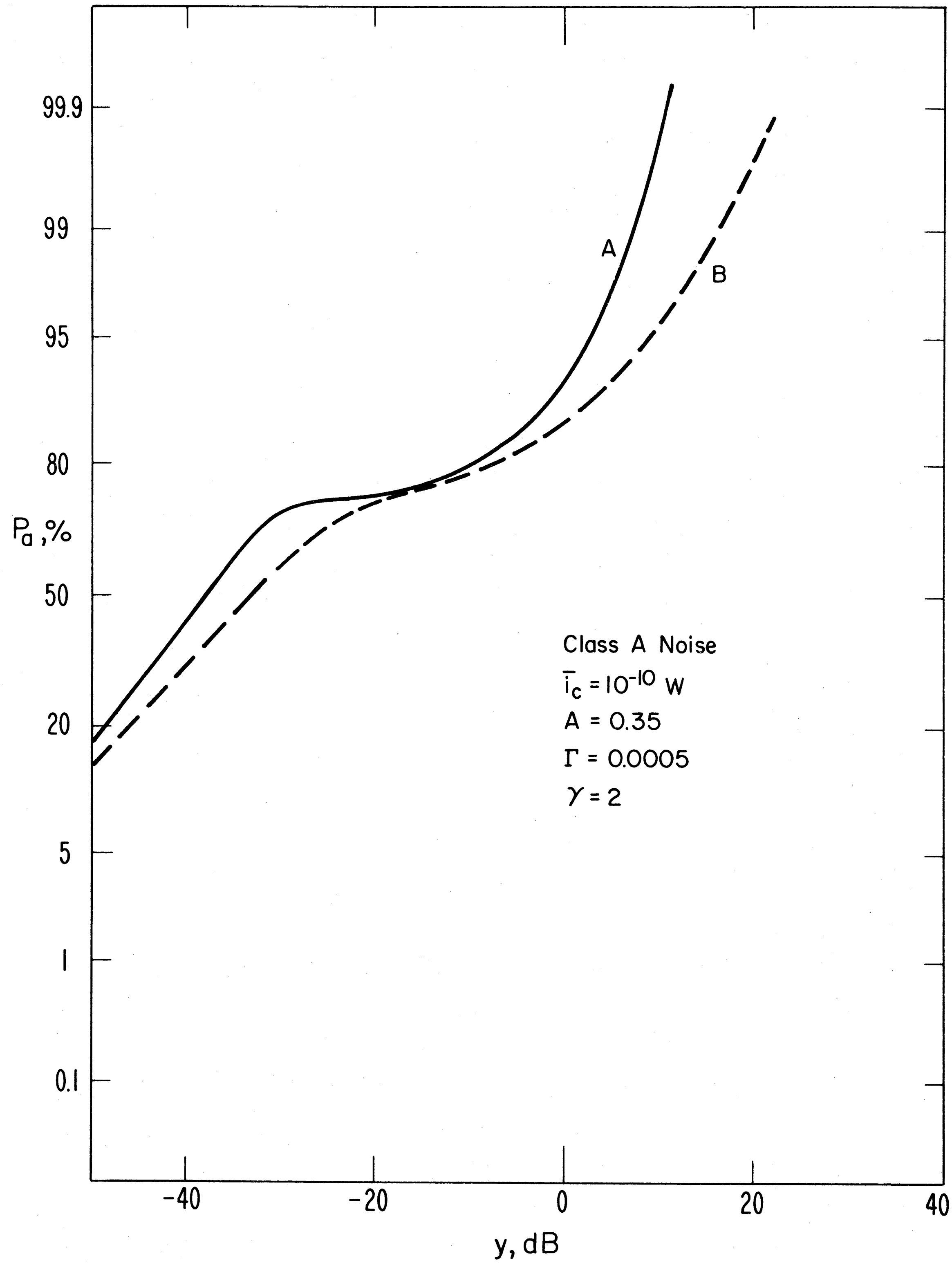

Figure 12. Probability of communications, P_a, for two different distributions of distance between the transmitter and receiver. The distance pdf used to compute curve A is given by (33) in the text; the distance pdf for curve B is given by (10).

4. RELATING CLASS A NOISE TO CONGESTED BAND SCENARIOS

The analytical solution for the probability of communications requires the three parameters of Class A noise as input. Middleton (1979a,b) suggests that these parameters must be determined by measurements and recommends measurement procedures. However, it will be several years (at least) before an adequate set of measurements is available. Another practical problem is that the probability of interference often must be computed before deploying new systems. In this case, it is impossible to measure the parameters of the Class A noise that will exist after the systems are deployed. So the ability to compute P_a from a specification of the characteristics and locations of interfering systems is necessary. This section explores the possibility of making this calculation with the analytical model.

4.1 Interpretation of Factors in Class A Noise Formula

Equation (22) gives the probability density function of noise (plus interference) for an undisciplined service for the same assumptions made by Middleton (1972, 1976, 1979a). For M = ∞, it is

$$f_I(i_n) = \sum_{m=0}^{\infty} \left[e^{-U} \frac{U^m}{m!} \right] f_{I|m}(i_n|m) \quad , \tag{34}$$

where U is the traffic intensity defined by equation (20). The term in the square brackets is the probability that there will be exactly m simultaneous interfering transmissions at any particular instant of time, and $f_{I|m}$ is the pdf of noise given that there are exactly m interfering transmissions.

Now recopy the formula for Class A noise, (2):

$$w_i(i_n) = \sum_{m=0}^{\infty} \left[e^{-A} \frac{A^m}{m!} \right] \frac{g_m}{\bar{i}_n} \exp(-i_n g_m/\bar{i}_n) \quad , \tag{35}$$

where A is the traffic intensity perceived in the receiver at the output of the IF filter. Notice the similarity of (34) and (35). If every transmission by an interferer is observed by the wanted receiver, then A = U, and the factor in square brackets in (35) is identical to the corresponding factor in (34). It is interesting to consider whether the other factor in (35) can be identified as the probability density function of interference power given that m transmitters are transmitting for some distribution of transmitters and some propagation law.

Consider the term for m=0. The factor multiplying the square brackets is

$$\frac{g_0}{\bar{i}_n} \exp(-i \frac{g_0}{\bar{i}_n}) \quad . \tag{36}$$

Using (3), (4), and (6), (36) can be reduced to

$$\frac{1}{\bar{n}} \exp(-i_n/\bar{n}) \quad , \tag{37}$$

which is recognized as the pdf of an exponential distribution with mean $\bar{n}$ (Zehna, 1970, p. 142). Whenever it is assumed that the pdf of background noise, $f_{I|0}$, is given by (37), then the first terms of (34) and (35) are identical.

For m=1, the coefficient of the square brackets in (35) can be shown to be a pdf of interference from one interferer. To do so, assume that the pdf of path lengths from interferers to the wanted receiver is given by

$$w_r(r) = 2\gamma \ p_1 \lambda r^{-2\gamma-1} \exp(-\lambda p_1/r^{2\gamma}) \quad , \tag{38}$$

where

$$\lambda = g_1/\bar{i}_n \quad . \tag{39}$$

Then, if the propagation law is given by equation (8), the interfering power from one interferer is

$$t = p_1/r^{2\gamma} \quad . \tag{40}$$

Applying the transformation in Table 2, the probability density of interference from exactly one interferer is

$$f_{I|1}(i_n|1) = \lambda e^{-\lambda i_n} = \frac{g_1}{\bar{i}_n} \exp(-g_1 \ i_n/\bar{i}_n) \quad , \tag{41}$$

which is the coefficient of the square bracket for the m=1 term in (35). Thus, the m=1 terms of (34) and (35) are the same provided that the probability density distribution of interferers is given by (38). This is the reason for choosing the form of (38). Middleton (private communication, 1980) has shown that the strictly canonical Class A noise model is exact when (38) is the pdf of source distance.

The spatial distribution of interferers is fixed by (38). If there are two interferers, the pdf of interference power from each is given by (41), and interference power from two interferers is the sum of two random variables with this pdf. Using the addition formula in Table 2, this pdf is

$$f_{I|2}(i_n|2) = \lambda^2 \, i_n e^{-\lambda i_n} \quad , \tag{42}$$

which cannot be reduced to the coefficient of the square bracket for m=2 in equation (35). Therefore, (34) and (35) cannot be equated term-by-term even under the most favorable assumptions.

Indeed, the fact that two infinite series have the same sum is never sufficient reason to expect the corresponding terms to be equal. However, the identification of the first two terms provides useful physical insight when A=U is small, because then the first two terms are a good approximation for the pdf except for the tail on the right end. In fact, for an example in Section 4.2 below, the interpretation makes (34) an excellent approximation for (35) over the entire interesting range.

Middleton (1974, 1976) made some mathematical approximations in the derivation of equation (2). A mathematical approximation to a solution for a given physical situation is equivalent to some modification of the physical situation. It is usually not possible to determine directly what the modification is, but identification of the approximate solution with a solution for a different physical situation throws some light on the effect of the approximation. This is the value of the partial identification above. Even though Class A noise does not arise from such a situation, the pdf is the same as if the interference was produced by equal-amplitude transmitters distributed according to (38).

4.2 Class A Parameters Using Scenario Simulation

The numerical model described in Section 2.2 computes the pdf of interference from a number of interferers as an intermediate result, using (22). If this pdf can be approximated by Class A noise for some set of parameters, and if these parameters can be determined, then the probability of noninterference, P_a, can be computed using Middleton's formula (11) instead of the slower numerical procedure. Of course some detail is lost, as shown in Section 3, but the results are accurate enough for some purposes.

As an example, the pdf of interference was computed using the numerical model for the following input assumptions:

The interfering transmitters are randomly located in a ring with inner radius of 0.1 km and outer radius 10 km with the victim receiver at the center of the circle. The probability that any small area contains an interfering transmitter is proportional to the size of the area. This implies that the pdf of path lengths from interferers to victim is

$$f_d(d) = \frac{2d}{10^2-(.1)^2} \quad . \tag{43}$$

The effective radiated power of interfering transmitters has a mean value of -40 dBW and is log normally distributed with a standard deviation of 3 dB. The transmission loss is normally distributed in decibels with a mean of -40 -40 log d (equivalent to γ=2) and a standard deviation of 5 dB.

The interfering transmitters transmit at random times, with traffic intensity U=0.25. The background noise power is normally distributed in decibels with a mean of -132 dBW and a standard deviation of 5 dB.

The series in (22) converged in six terms. Figure 13 shows the pdf of noise plus interference for this case. The dashed line shows the interference plus noise when only one interferer is transmitting. Clearly, there is more than one interferer for only a small fraction of the time, because the total interference is only slightly more than that for one interferer. Mathematically, the coefficients of the higher-order terms for U = 0.25, given in Table 3, are about an order of magnitude smaller than the coefficient for m=1.

The curve in Figure 13 looks generally like the one for Class A noise in Figure 4. To use formula (11), the Class A noise parameters A, Γ, and $\bar{i}_n$ must be known. Assuming that all interfering transmissions are perceptible in the receiver, A=U.

One way to determine Γ and $\bar{i}_n$ is based on the partial identification at the end of Section 4.1--that (34) and (35) can be approximately equated term by term. Substituting (3) into (30) for m=0 and m=1, yields

$$I_N(0) = b_o = 10 \log_{10}\left(\bar{i}_n \frac{\Gamma}{1+\Gamma}\right) \quad , \tag{44}$$

and

$$I_N(1) = b_1 = 10 \log_{10}\left(\bar{i}_n \frac{1+\Gamma}{A+A\Gamma}\right) \quad . \tag{45}$$

Recall that $I_N(m)$ is the location of the maximum of the pdf for the specified value of m, which can be read from the curve in Figure 13. So there are only two unknowns in (44) and (45). These equations can be written:

$$10^{b_o/10} = \bar{i}_n \frac{\Gamma}{1+\Gamma} \quad , \tag{46}$$

and

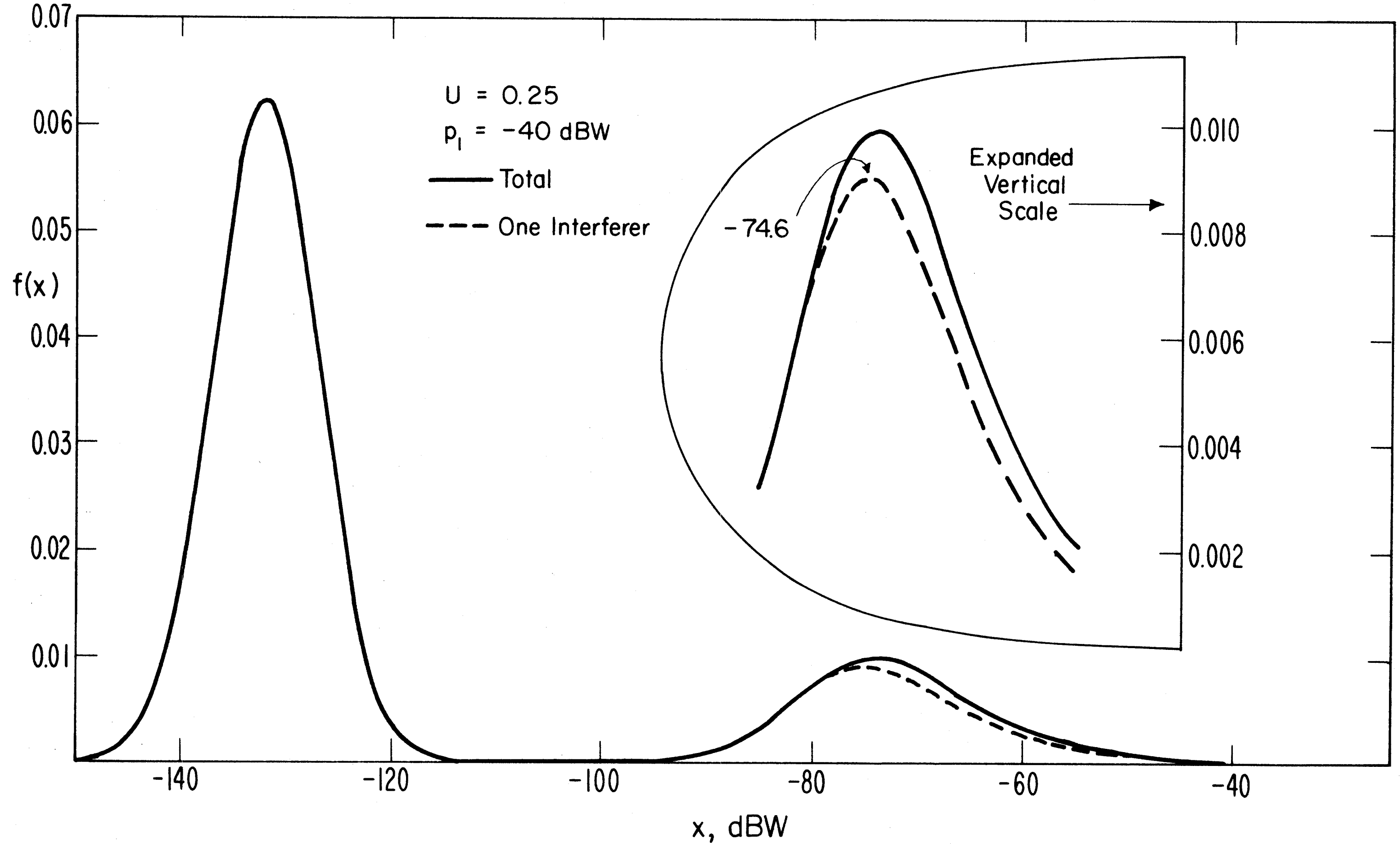

Figure 13. The probability density function of noise-plus-interference for the first scenario in Section 4.2. The inset is the interference portion of the pdf with an expanded vertical scale. Most of the interference is due to one simultaneous interferer.

$$10^{b_1/10} = \bar{i}_n \frac{1+\Gamma}{A+A\Gamma} \quad . \tag{47}$$

Dividing (46) by (47), eliminates $\bar{i}_n$, and

$$\Gamma = \frac{10^{b_o/10-b_1/10}}{A\left(1-10^{b_o/10-b_1/10}\right)} \quad . \tag{48}$$

Then

$$\bar{i}_n = (1-A)\ 10^{b_o/10} + A10^{b_1/10} \quad . \tag{49}$$

From Figure 13, $b_o = -132$ and $b_1 = -74.6$, so $\Gamma = 7.28(10^{-6})$ and $\bar{i}_n = 8.67(10^{-9})$. Figure 14 compares the Class A noise pdf for these parameter values with the pdf of noise and interference computed with the numerical model. The dashed line in Figure 14 is copied from Figure 13 for easy comparison.

The Class A peak near -75 dBW is much higher and narrower than the peak computed from the scenario (dashed line). If the conjecture in Section 4.1 is accepted for the moment, this difference is probably due to the difference between the geographic distribution of interferers assumed in the computation as given by (43) and the distribution implied by the form of Class A noise, as given by (38). These two different pdfs of interfering path lengths are shown in Figure 15. Curve B is the path-length pdf given by (38) for this scenario, as discussed in Section 4.1. Because there are almost no interferers closer than 4 km for Curve B, there is virtually no probability of interference stronger than -65 dBW in Curve A of Figure 14. However, for Curve A of Figure 15, there is substantial probability of interferers located .1-4 km from the receiver. This explains the long tail above -65 dBW on the scenario pdf in Figure 14. The curve-fitting procedure forces both pdfs to have the same area under the peaks on the right, so the narrower peak must be higher.

At this point it is possible to test the accuracy of the interpretation of Class A noise given in Section 4.1. Equation (34) (the computer model) was evaluated with the same parameters as in the scenario immediately above, except that the pdf of interfering path lengths was Curve B of Figure 15 instead of Curve A; that is, the pdf implied by the interpretation of the m=1 curve was used. (In addition, the standard deviation of transmitter power and the standard deviation of transmission loss were both reduced to 1 dB to simulate the deterministic nature of

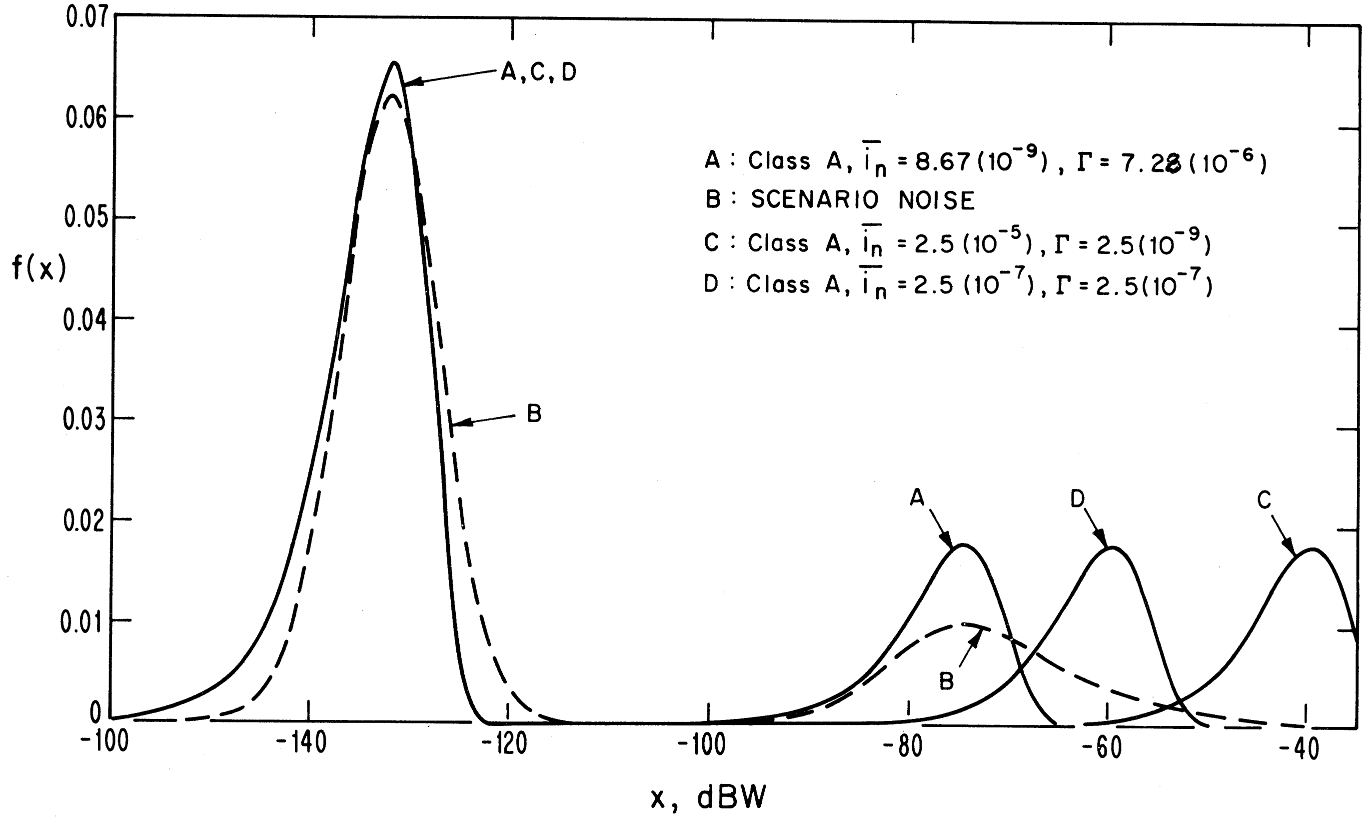

Figure 14. Comparison of the scenario pdf (Curve B) of Figure 13 with the best-fit Class A noise approximation (Curve A). The scenario has higher probability of strong interference. Curves C and D are fits for different values of $\overline{T}_o$.

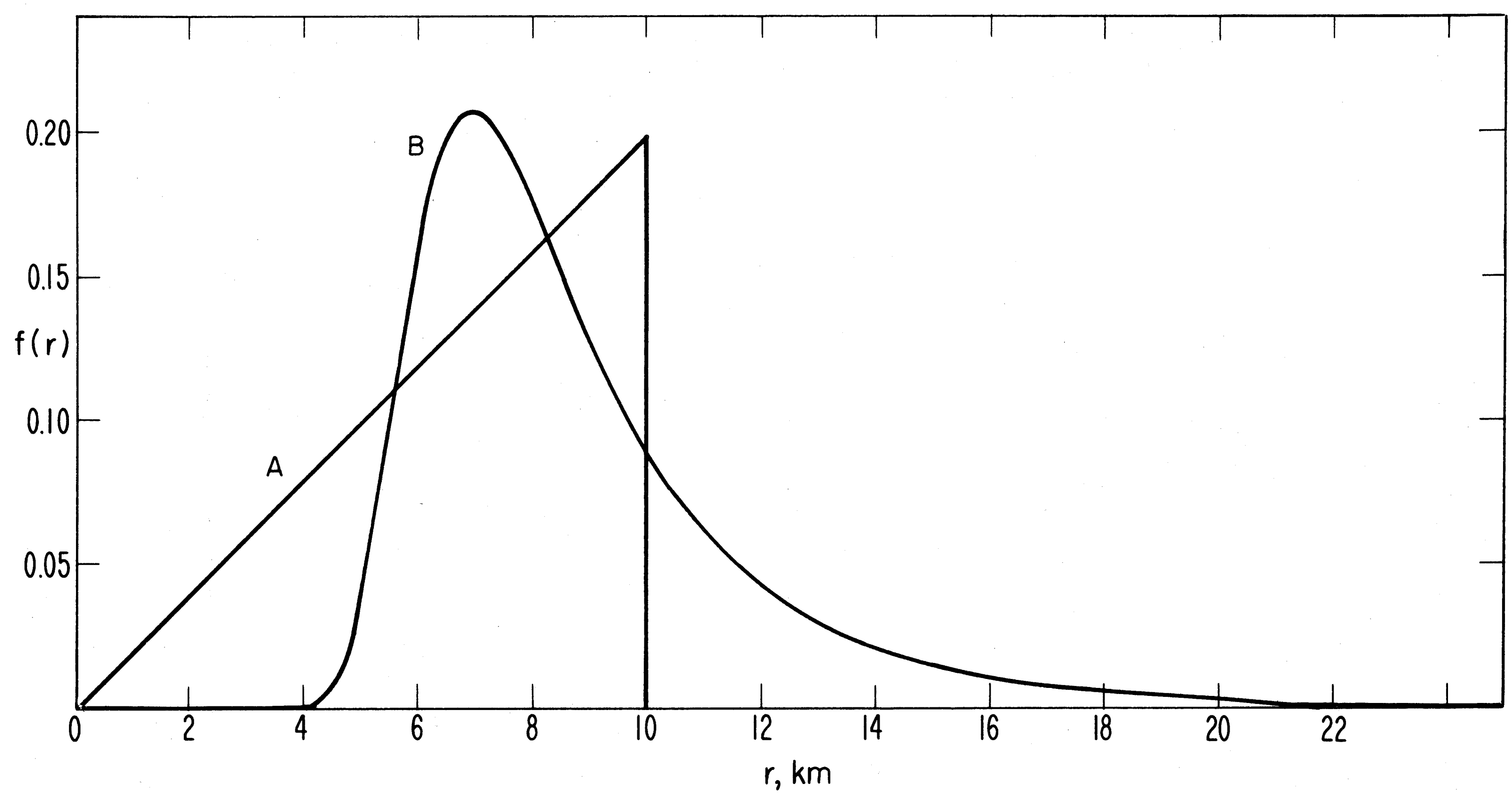

Figure 15. Comparison of the pdf of interfering path lengths in the scenario (Curve A) and the pdf of interfering path lengths implicit in the m=1 term of Class A noise (Curve B), for the best-fit curve of Figure 14. Curve B shows very small probability of paths shorter than 4 km.

these quantities in the derivation of Class A noise.) The right (interference) bump exactly matches that of Class A noise in Curve A of Figure 14.

Another way to determine Γ and $\overline{i}_n$ is suggested by Appendix A-I of Middleton's report (Middleton, 1979a). The mathematical details of this procedure are in the Appendix of this report; the result for the scenario described at the beginning of this section is

$$\overline{i}_c = \langle e_o^2 \rangle \; p_1' (d_s d_m)^{-2} \; A \quad , \tag{50}$$

where

e_o is a "limiting sensitivity voltage," and <> means average;
p_1' is a normalized power of an interfering source;
d_s is the shortest distance to an interferer; and
d_m is the maximum distance to an interferer.

In the present scenario, $d_s = 0.1$, $d_m = 10$. I believe that if $p_1' = p_1$ (Table 1) then $\langle e_o^2 \rangle = 1$, so $\overline{i}_c = A\,(10^{-4}) = 2.5(10^{-5})$. Then $\Gamma = \overline{n}/\overline{i} = 10^{-13.2}/\overline{i}_c = 2.52(10^{-9})$. Curve C of Figure 14 shows the Class A noise pdf for this set of parameters. The interference bump is the same size and shape as for Curve A, but is much further to the right--centered at -40 dBW.

Perhaps the product $\langle e_o^2 \rangle \; p_1'$ has been incorrectly interpreted. As a test, the pdf was computed for $\langle e_o^2 \rangle \; p_1' = 10^{-6}$, so that $\overline{i}_c = 2.5(10^{-7})$. Curve D of Figure 14 is the Class A noise pdf for this set of parameters. The only change is that the interference bump is now centered over -60 dBW.

By now it should be clear that as long as A=0.25 and $\Gamma \ll 1$, the interference bump (nongaussian noise) will have the same size and shape. Changing $\overline{i}_c$ only translates the bump along the abscissa.

In the comparisons above, A=U, and the fits are unsatisfactory. Figure 16 shows several heuristic attempts to achieve a better fit by changing A. Curves (1), (2), and (3) were attempts to get more probability of interference at levels above -65 dBW and to reduce the right peak of Class A noise to about the same height as that of the scenario noise. The result is a poor fit to the left of -75 dBW, without achieving the required probability to the right of -60 dBW.

The parameters for curves 4 and 5 keep the maximum of the right peak at the correct location and bring its height down nearer the height of the scenario interference. These curves do not have sufficient probability of interference above -65 dBW.

Curves (1) through (5) in Figure 16 suffer from the defect A≠U, which is the one parameter known from the scenario description. This results in insufficient

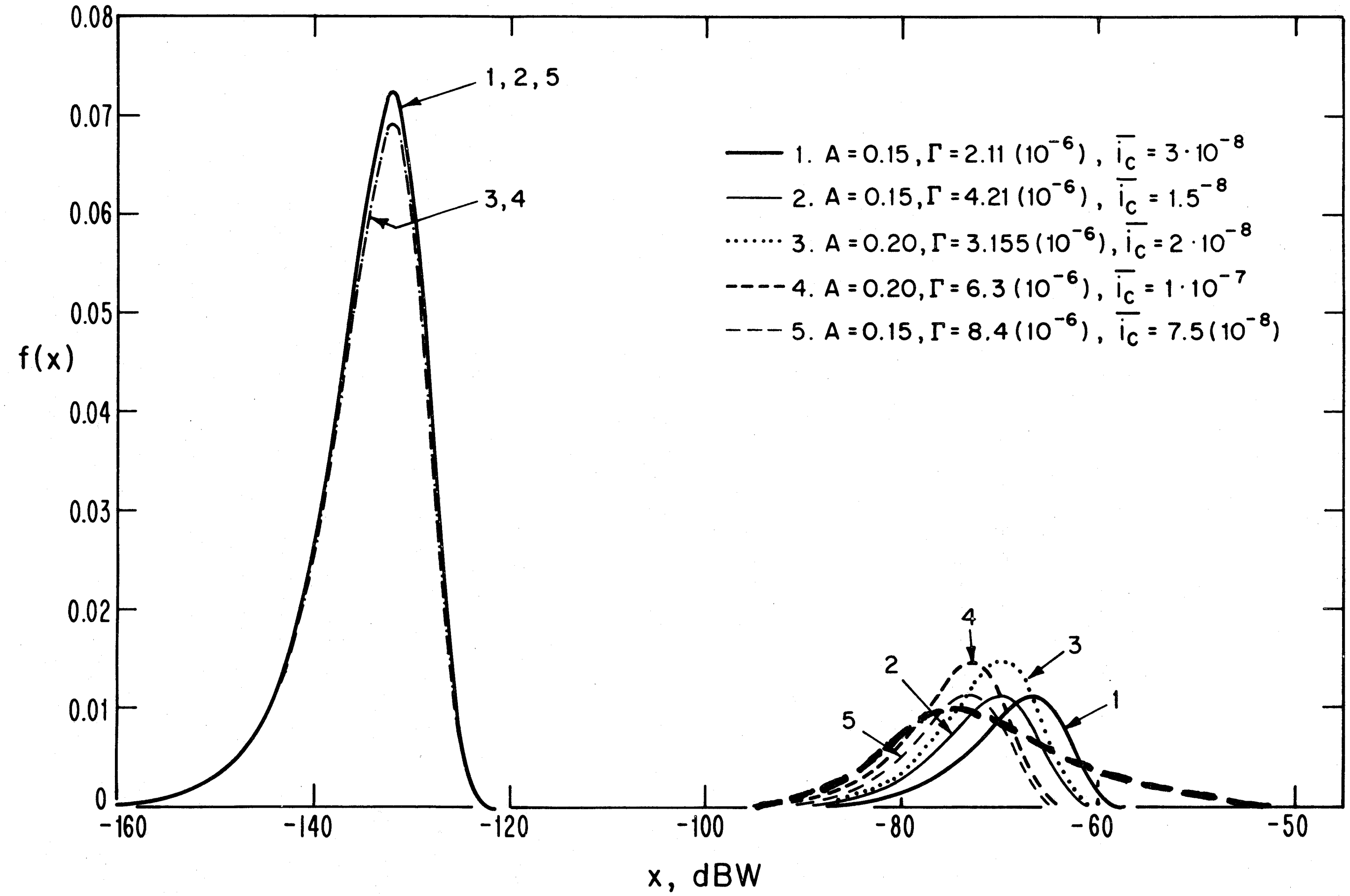

Figure 16. Class A noise intensity pdfs for several sets of parameters. The curves are attempts to fit the scenario interference which is shown as the heavy dashed line.

area under the right bump in all the curves; that is, the probability of interference from gaussian noise is not large enough.

Study of Figures 14 and 16 makes it hard to imagine any set of parameters that would make Class A noise fit the pdf of Figure 13. The basic problem is that the Class A nongaussian interference bump has about the same shape for all reasonable values of the parameters, when a different shape, with a long tail to the right, is needed. There is no parameter in the Class A noise formula that can be used to spread the pdf of nongaussian interference. Perhaps this should have been anticipated, because the Class A parameters involve only the means of the two kinds of noise and not their variances.

It should be noted that (2) is an approximation to the pdf of the envelope of Class A noise (Middleton, 1976). The more exact form includes terms that extend the tail to the right. However, no analytical solution for P_a exists for this form; so its consideration at this time is fruitless.

Of more practical importance than the pdf of noise is the probability of communications for a given scenario. Figure 17 compares P_a, the probability that the signal-to-noise ratio exceeds the required threshold for noise curves A and B in Figure 14. In both cases, the pdf of distance of the wanted transmitter from the receiver is given by Curve A of Figure 2. Transmission loss is given by (8) with γ=2; it has no location variability. The dashed line in Figure 17 is P_a for the scenario interference; the solid line is P_a for the Class A noise. The two curves are fairly close together for P_a less than 70%--that is, until the interference becomes important and the difference between the two bumps on the right in Figure 14 becomes important. In this region, the calculation using the Class A noise approximation consistently overestimates the probability of communications. Looked at another way, if a reliability P_a = 99% is required, the Class A approximation underestimates the required transmitter power (represented by y) by 14 dB. If a reliability of 90% is required, the approximation underestimates the required transmitter power by 5 dB.

Figure 18 shows noise plus interference for a scenario with less separation between noise and interference. The geographic distribution of interferers is the same as for Figure 13. The traffic intensity U=0.5 and the reference power is p_1 = -85 dBW. The dashed line in Figure 18 is the pdf of interference from a single interferer. Using the locations of the two peaks, the parameters for a Class A noise model can be found using (48) and (49). They are Γ = 0.137 and $\bar{i}_n$ = 5.31 (10^{-13}). Figure 19 compares the Class A noise for these parameters with the noise from the scenario. As in the previous case, the nongaussian Class A noise has a higher peak than the interference from the scenario, and the scenario has a higher

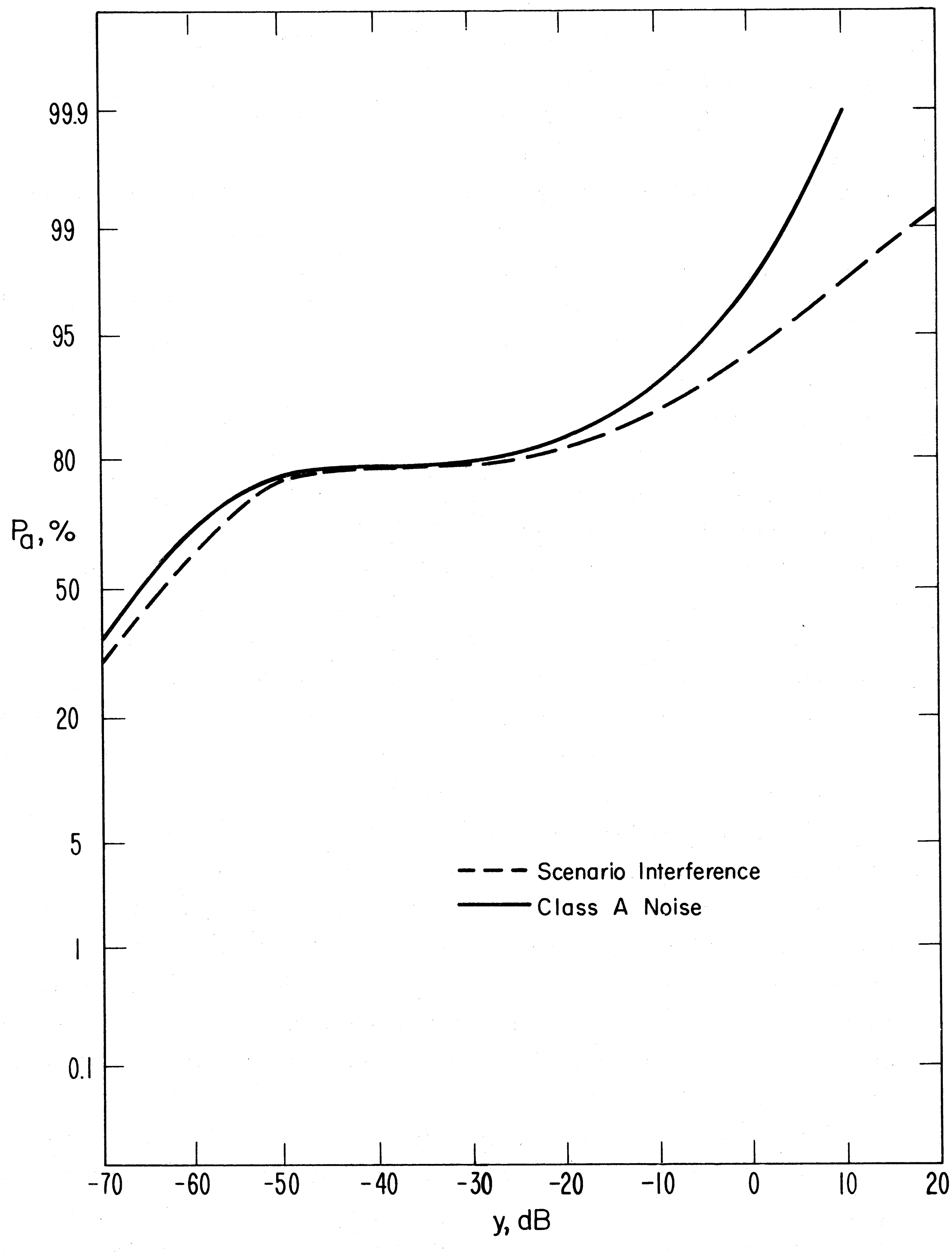

Figure 17. Comparison of the probability of communications, P_a, for the first scenario (dashed line) with P_a computed using the best-fit Class A noise.

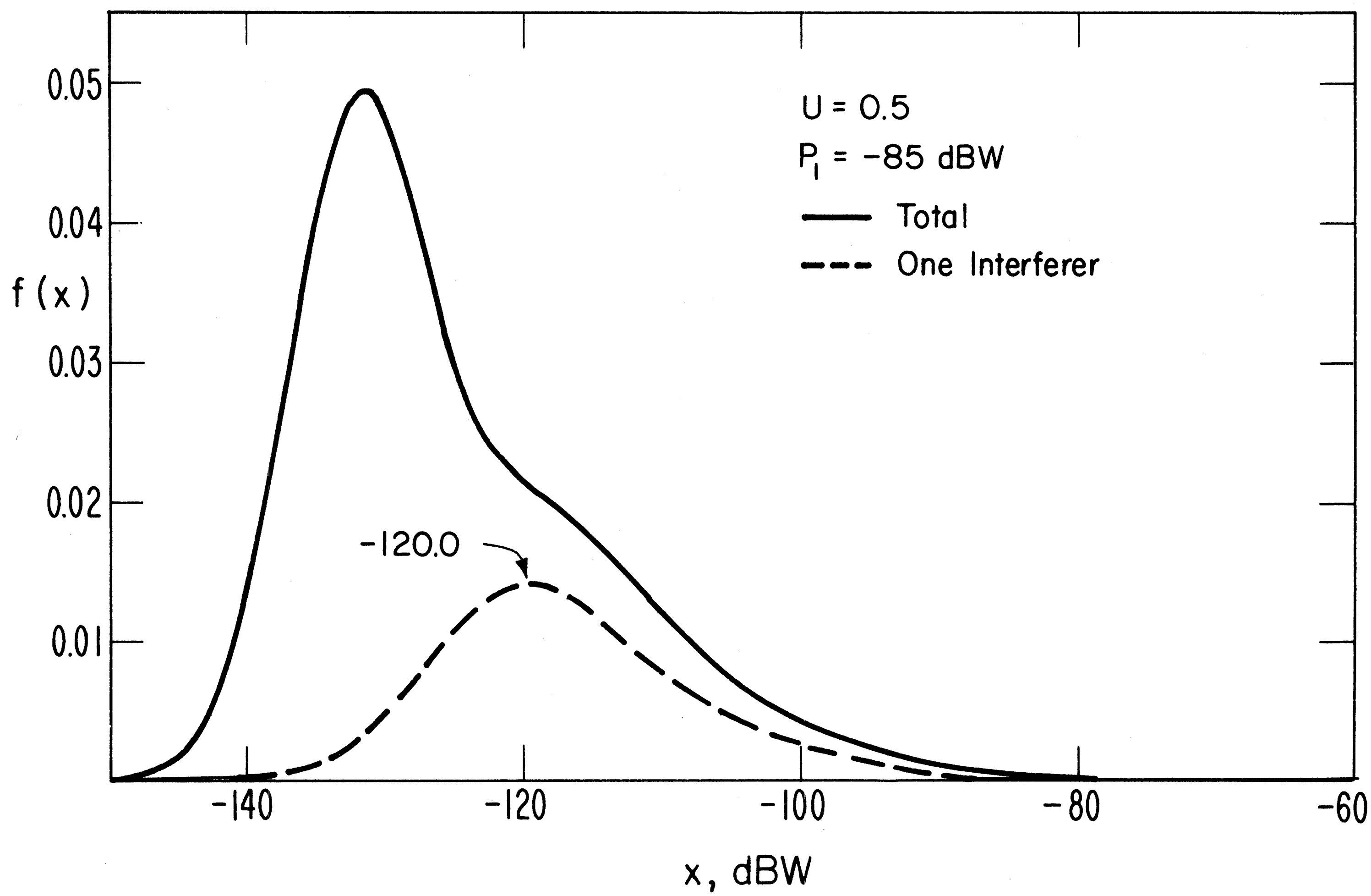

Figure 18. The probability density function of noise-plus-interference for a scenario with interference only slightly stronger than the background noise.

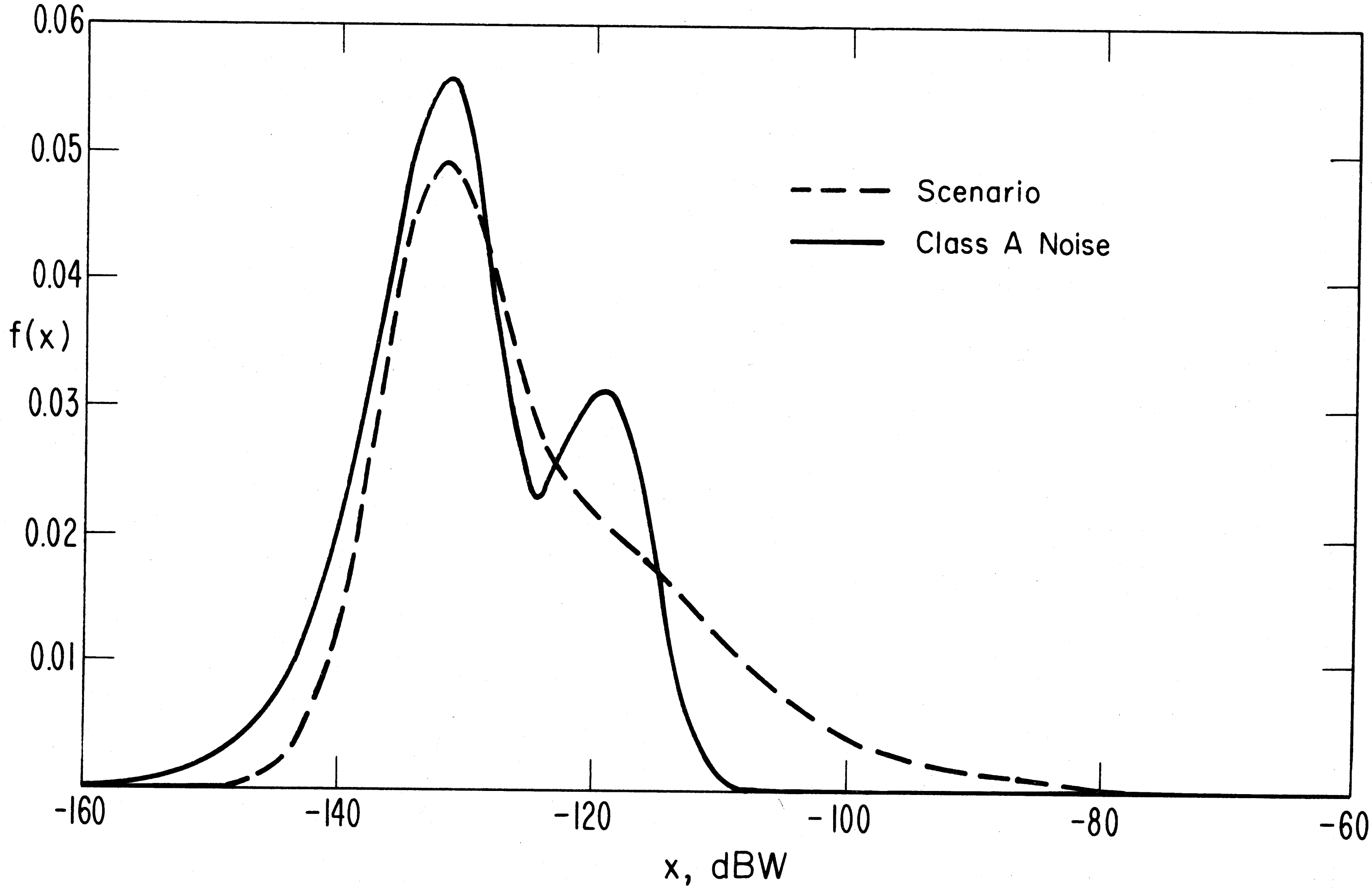

Figure 19. Comparison of pdfs of noise-plus-interference from second scenario (dashed line) with best-fit Class A noise (solid line).

probability of interference stronger than -110 dBW. As before, it is unlikely that a better fit can be achieved because there is no way to spread out the nongaussian bump in Class A noise.

Figure 20 shows the probability of communications for both models of noise in Figure 19. Again, the calculation using Class A noise overestimates the probability of communications for a given transmitter power. For 99% reliability, the required power is underestimated by 18 dB; for 90% reliability, it is underestimated by 8 dB.

Recall that the point of this section was to see if Class A noise parameters can be determined from calculations of the noise-plus-interference for an arbitrary scenario. The possibility of doing this approximately has been illustrated for two cases. It is possible to define scenarios that produce much closer fits. For example, if, in the scenario at the beginning of this section, the pdf of path lengths is changed from (43) to (38)--that is, from Curve A to Curve B in Figure 15, then the calculated pdf of noise plus interference exactly matches the Class A noise pdf of Figure 14. On the other hand, using the insights gained from these examples, it would be possible to define scenarios that result in interference pdfs much different from the Class A pdf.

5. THE NUMBER OF CO-CHANNEL LINKS THAT CAN OPERATE IN A REGION

How many co-channel transmitter-receiver (TR) links can achieve a specified reliability in a given geographic region? This question is basic to spectrum management because its answer defines the assignment capacity of a frequency band and area. The answer is useful only if the transmitters are confined to a specified area or if the geographic density is given. (It is clear without making any calculations that an infinite number of TR links can operate in an infinite area.)

This question can be answered using the computer program described in Section 2.2. Assume a scenario in which all links are statistically similar to the one described at the beginning of Section 4.2, except for the traffic intensity, U. Instead assume that the average traffic intensity for each individual TR link is U'. Then the total traffic intensity in the region is

$$U = M\,U' \quad ,$$

where M is the number of links in the region.

Figure 21 shows the pdf of noise and interference for $U' = 0.1$ and different values of M. Figure 22 shows the probability that the signal-to-noise ratio exceeds R as a function of R. Figure 22 can be replotted to show the same probability as a function of M, for fixed R. Suppose that the required signal-to-noise ratio for a

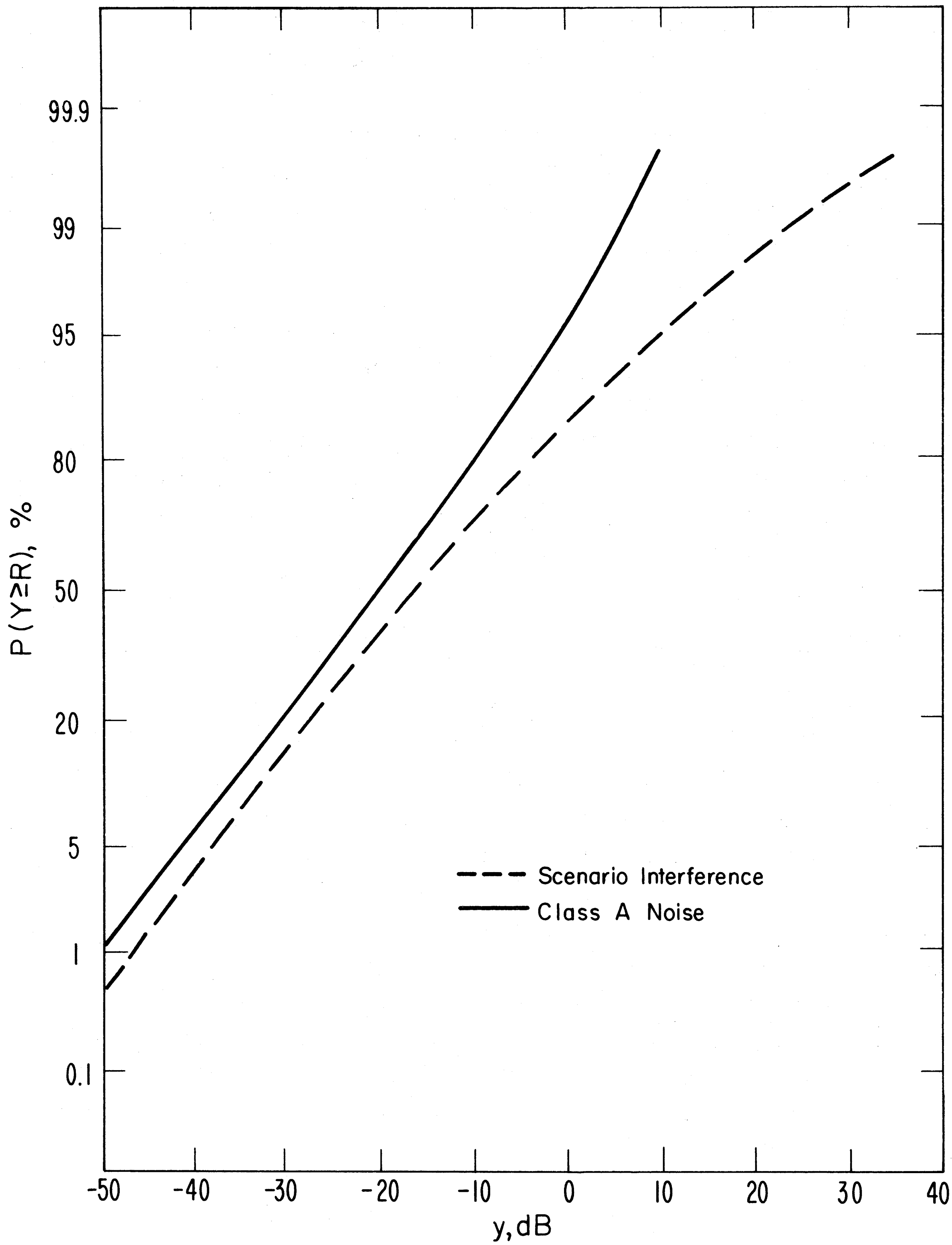

Figure 20. Comparison of the probability of communications, P_a, for the scenario (dashed line) with P_a computed using the best-fit Class A noise.

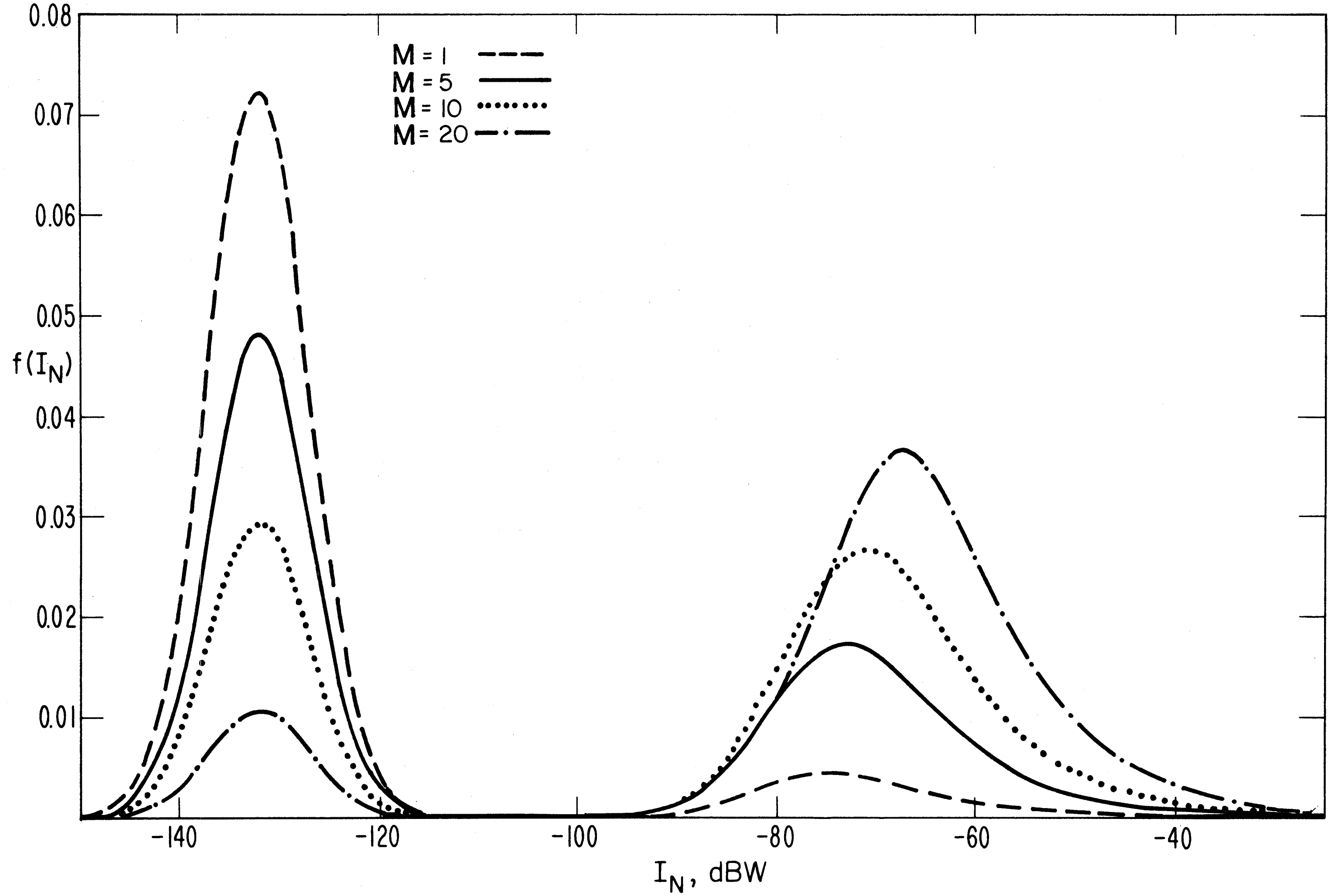

Figure 21. Probability density function of noise-plus-interference for different numbers of TR links assigned in a circle of radius 10 km. Each link has a traffic intensity U'=0.1.

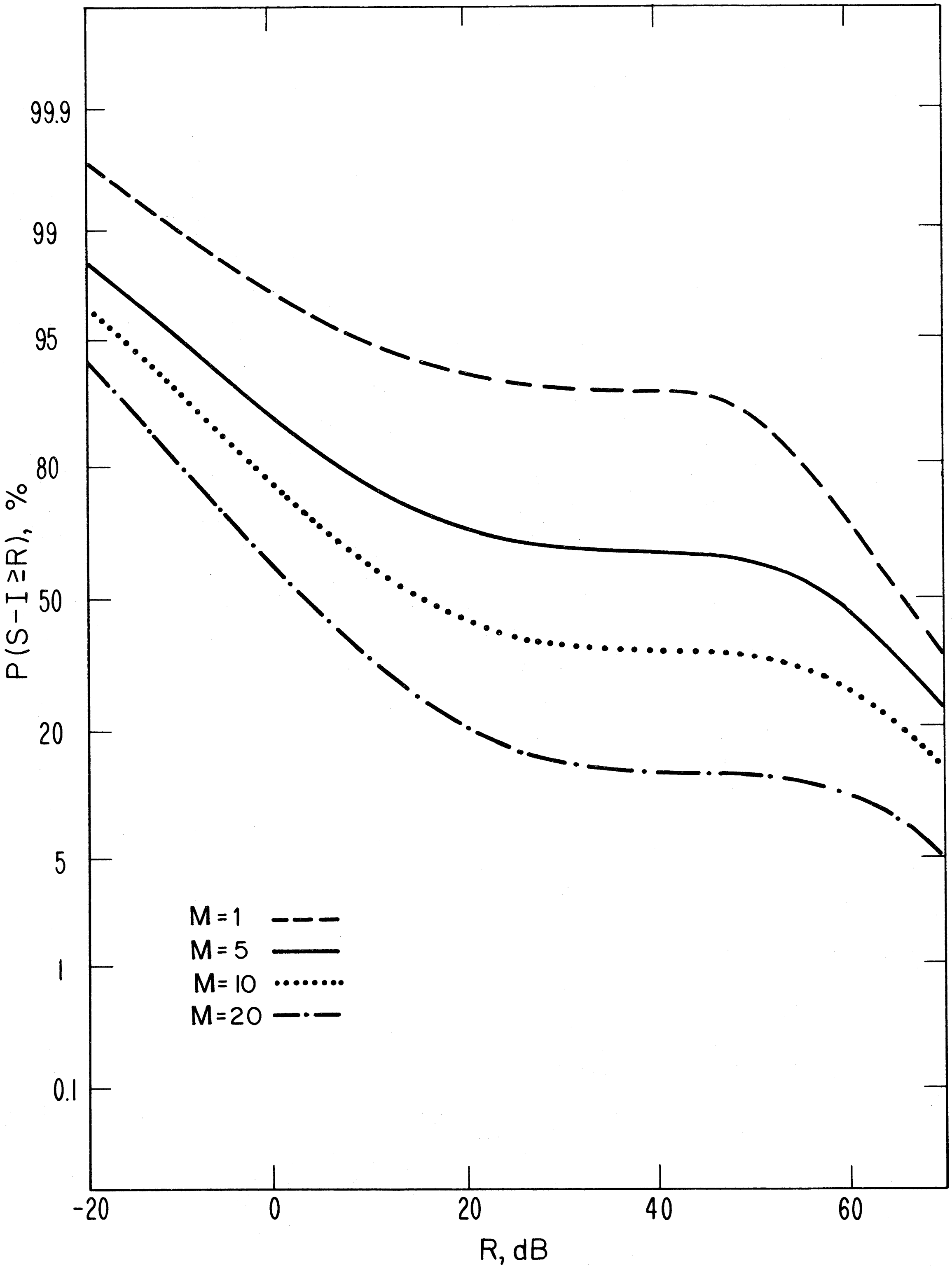

Figure 22. Probability that the signal-to-interference ratio exceeds a threshold R, for the scenarios of Figure 21.

service is R=10 dB. Figure 23 shows the probability that this is achieved as a function of M, the number of links in the area. If the reliability that the average link is to achieve is specified, then the number of links that can operate with this reliability can be read from the curve. For example, if the desired reliability is 75%, the number of links is about M=5. If the desired reliability is 90%, the number of links that can operate satisfactorily is only about M=2.

The analytical solution for P_a, equation (11), cannot be used alone for determining the number of links that can operate satisfactorily in a given area because the source distribution and density do not appear in the formula. However, if the source distribution is one for which the canonical Class A noise model is a good approximation, then a procedure similar to that used by Middleton (1978a, Section 5.4) could be used. An acceptable source distribution is Curve B of Figure 15 (equation 38) or a distribution similar to it. Note that the uniform source density distribution used in Section 4.2 does not approximate (38).

The answer to this section's question that was found with the numerical model must be evaluated in the light of the assumption made about operating procedure. For direct comparison with Middleton's model, it was assumed that transmitter turn-ons were independent--that is, that there was no circuit discipline or courtesy. This assumption, which is basic to Class A noise (Middleton, 1972, 1976), is perhaps a good approximation for the Citizens Band radio service, but is probably not valid for any other service. However, the assumption of independent turn-ons is not necessary to the numerical model. All that is necessary to make a more realistic calculation of the assignment capacity with the numerical model is to replace equation (19) for P(m on) with a more realistic formula. The method followed above will then produce the desired result.

6. CONCLUSIONS

The conclusions in this section apply only to the available analytical solution, (11) (Middleton, 1979a), and the available computer program (Berry, 1977). These are the tools which a frequency manager can use to get immediate quantitative results. The two models produce identical output for identical input, as shown in Section 2.3.

Table 4 summarizes the essential assumptions for both models, the required and optional input, and the normal output.

It is clear from the input list in Table 4 that the numerical model has more flexibility than the present analytical solution, equation (11), but is computationally slower. The analytical solution was derived from a formal solution with flexibility comparable to the numerical model, but the formal solution has not been programmed, and its programming would be nontrivial. Indeed, such a program would be very similar to the already-available program for the numerical model.

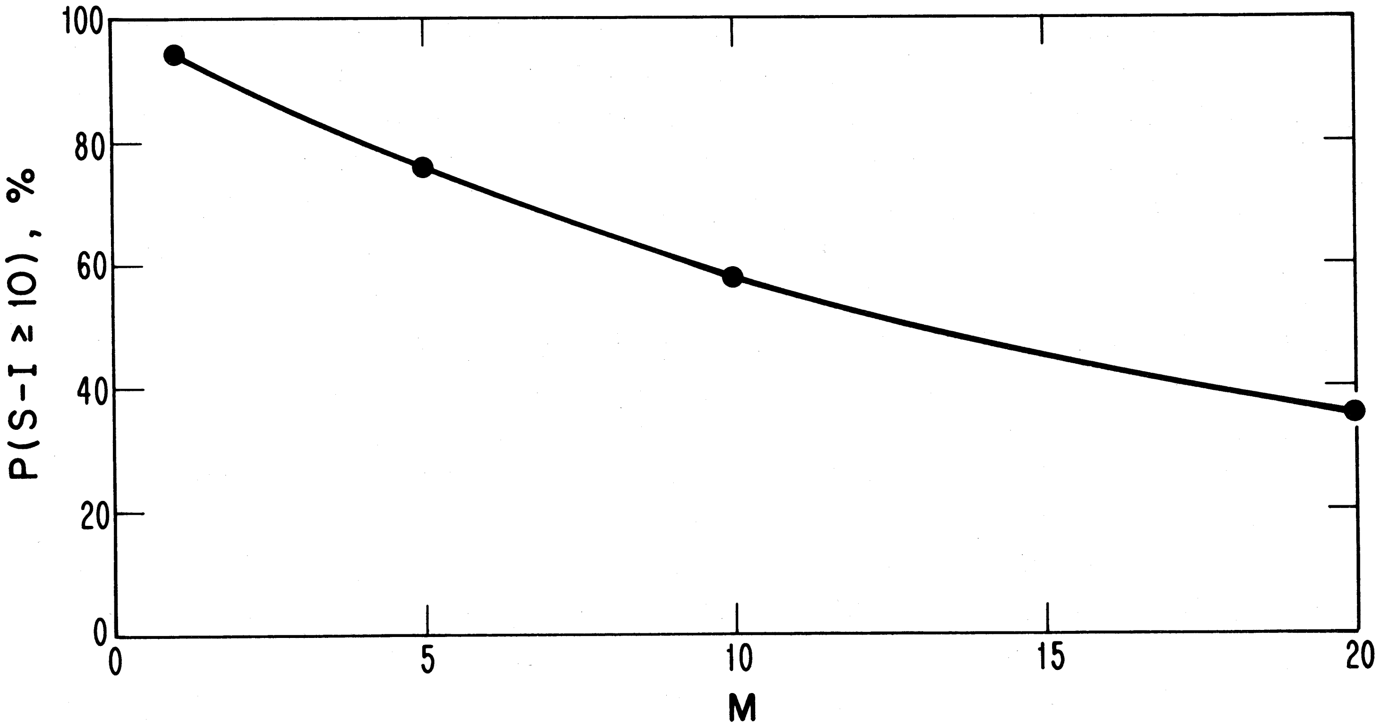

Figure 23. Probability that the SNR exceeds 10 dB as a function of the number of potential interferers, M. The points are taken from Figure 22.

Table 4. Comparison of the Available Analytical Solution for $P(s/i \geq r)$ (Middleton, 1979a)[1] and the Available Computer Program for the Same Probability (Berry, 1977)

	ANALYTICAL SOLUTION	NUMERICAL COMPUTER PROGRAM
OUTPUT	Probability that the signal-to-noise ratio exceeds the required threshold.	Probability that the signal-to-noise ratio exceeds the required threshold.
DESIRED LINK INPUT	Transmitter power is deterministic. Input parameter is power/(required s/i)/(mean noise).	Transmitter power is probabilistic. Input is pdf of transmitter erp specified analytically or in a table.
	Propagation is deterministic with received power proportional to $1/d^{2\gamma}$, where d is path length. Input parameter is γ.	Propagation is probabilistic. Input is conditional pdf of transmission loss given path length, specified analytically or numerically.
	Desired path length is probabilistic. Path length pdf has the specific form in equation (10). Input parameters are the scale factor d_0 and the distance between the receiver and the center of operations of the transmitter.	Desired path length is probabilistic. Input is arbitrary pdf of wanted path length.
	Required signal-to-noise ratio, s/i. Input parameter is relative to transmitter power, power/(required s/i)/(mean noise).	Required signal-to-noise ratio is an input parameter.

[1]The limitations of the analytical solution, (11), are not intrinsic to Middleton's formal solution. The formal solution is capable of sophistication comparable to that of the numerical model when the quasi-canonical Class A model is introduced (Middleton, private communication, 1980), although reduction of the formal solution to an analytical solution is still required.

Table 4. (continued)

	ANALYTICAL SOLUTION	NUMERICAL COMPUTER PROGRAM
INTERFERING LINKS INPUT	Interference environment is described by Class A noise. Input parameters are: A, the overlap index (traffic intensity). $\bar{i}_n$, the mean noise. Γ, the ratio of the mean of the gaussian noise to the mean of the nongaussian noise. Underlying assumptions are that there are many potential sources, poisson distributed in space, and transmitting independently (no circuit discipline or courtesy).	Interference environment is the sum of interfering power from all categories of interferers. For each category of interferer: Transmitter power is probabilistic. Input is pdf of erp. Propagation is probabilistic. Input is conditional pdf of transmission loss, given distance. Interfering path length is probabilistic. Input is pdf of path lengths. Operational procedures (circuit discipline) and traffic intensity are combined into the probability that m interferers are emitting simultaneously, which is input. Optional: Receiver transfer function (e.g., frequency dependent rejection) can be specified for this category of interferer. There can be several categories of interferers. A different category of interferer is defined if one of its specified pdf's is different from other categories. Ambient noise can be separately specified. These inputs allow interference to be calculated from a detailed scenario.

There is an intriguing similarity between the series for the pdf of Class A noise (2) and the pdf of background noise and interference from an infinite number of interference sources which transmit independently. Under appropriate assumptions, the series have the same sum, the first terms of the two series are identical, and all other corresponding terms contain an identical factor. These conditions are not sufficient for a term-by-term identification, but the identification is useful for determining approximate Class A parameters from a scenario, as shown in Section 4.2.

The Class A noise parameters required by equation (11) can be approximately determined from calculations of signal and interference for a scenario. How well the approximation fits the scenario noise depends on how closely the scenario conforms to that implied by the interpretation discussed in the preceding paragraph. For the example in Section 4.2, the approximation overestimated the probability of communications.

A basic assumption in the derivation of the Class A noise pdf is that interfering sources transmit randomly. In land-mobile radio terms, this assumption means that there is no circuit discipline--no operator courtesy. The numerical model does not require this assumption. Any degree of circuit discipline can be modeled by specifying the probability that m sources are transmitting for m=0, 1, 2, ...

The numerical model can be used to compute the number of links that can operate with specified reliability in a given area. The analytical solution, (11), can be used for this calculation for only those scenarios whose interference source distributions are approximated by (38).

7. ACKNOWLEDGMENTS

Dr. David Cohen of NTIA/Annapolis stimulated this study and contributed comments and information during it. Dr. David Middleton introduced me to his valuable work and explained some of the difficult points to me. His rigorous review of the first draft of this report resulted in substantial revision. Dr. A. D. Spaulding of NTIA/ITS provided computer programs which evaluate the analytical formula for S/N probability.

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APPENDIX: DIRECT CALCULATION OF CLASS A PARAMETERS FROM SCENARIO DESCRIPTORS

In Section 4.2, Class A noise parameters are determined for a scenario by fitting the pdf of interference computed numerically. In a private communication, Middleton has suggested that the parameters can be computed directly from scenario descriptors. In this appendix, this direct calculation is attempted, and the failure of the attempt is explained.

The scenario is the same as that given at the beginning of Section 4.2, which led to the interference pdf shown in Figure 13. For this case, A=U=0.25, and the mean of the background noise (-132 dBW) can be taken for the mean of the gaussian noise. The mean of the interference must be computed. Middleton (1979a) gives these relationships (converted to the notation of this report):

$$\overline{i}_c = A\langle B_o^2\rangle/2 \quad , \qquad \text{(A-1)}$$

and

$$\langle B_o^2\rangle = \langle e_o^2\rangle\langle u_o^2\rangle\langle A_o^2\rangle\langle |A_{RT}(\theta)|^2\rangle \; \langle g_s^2/(4\pi\, c\lambda)^{2\gamma}\rangle \quad , \qquad \text{(A-2)}$$

where

B_o	is the basic envelope of the interference measured in the receiver after the IF filter;
e_o	is the limiting sensitivity voltage of the receiver;
u_o	is the normalized interfering signal waveform;
A_o	is the peak amplitude of the interfering signal waveform;
$A_{RT}(\theta)$	is the combined antenna patterns of the receiver and transmitter;
g_s	is a constant when the distribution of interferers does not depend on azimuth around the receiver; and
$c\lambda=d$	is the distance of an interferer from the receiver (c is the speed of light).

As usual, the notation <> indicates an average. (Middleton (1979a) inadvertently omitted the average over the factor on the right in A-2.)

For the scenario in Section 4.2, $A_{RT}(\theta) = 1$ and $\gamma = 2$. Clearly, $\langle u_o^2\rangle\langle A_o^2\rangle/2 = p_1'$ is a power associated with the interferers (who all have the same power). Consistent with earlier usage, it is assumed that these quantities are measured 1 km from the transmitter so that $p_1' = p_1$, the effective radiated power.

For the distribution of interferers in the scenario, d has the probability density function (pdf)

$$f_d(d) = \frac{2d}{d_m^2 - d_s^2} \quad , \text{ for } d_s \leq d \leq d_m \quad . \tag{A-3}$$

For the specific scenario of interest, $d_m = 10$ and $d_s = 0.1$.

Now that d is random, (A-2) must be modified so that the average of $1/(c\lambda)^{2\gamma} = 1/d^4$ is taken. Let $v = 1/d^4$. Then the pdf of v is found using the transformation in Table 2 of the report. It is

$$f_v(v) = \frac{v^{-3/2}}{2(d_m^2 - d_s^2)} \quad , \quad v_o \leq v \leq v_m \quad , \tag{A-4}$$

where $v_o = d_m^{-4}$ and $v_m = v_s^{-4}$. Then the expected value of v is

$$E(v) = \int_{v_o}^{v_m} v f_v(v)\, dv = \frac{1}{2(d_m^2 - d_s^2)} \left[2\ v^{\frac{1}{2}} \right]_{v_o}^{v_m} \quad . \tag{A-5}$$

So

$$E(v) = (d_s d_m)^{-2} \quad . \tag{A-6}$$

This is the value Middleton derives in a slightly different way in his private communication. Inserting this value and the other values determined above into (A-2) yields

$$\langle B_o^2 \rangle = (\langle e_o^2 \rangle\ g_s^2/(4\pi)^2)\ 2\ p_1/(d_s^2\ d_m^2) \quad . \tag{A-7}$$

The parameters e_o and g_s are not given in the scenario, but for now take

$$g_s = 4\pi \quad , \tag{A-8}$$

so that

$$\langle B_o^2 \rangle = \langle e_o^2 \rangle\ 2p_1/(d_s^2\ d_m^2) \quad . \tag{A-9}$$

This is the result used in (50) of the report. It is clear from the discussion following (50) that the assumption in (A-8) is not critical. It is also clear from this discussion that direct calculation of Class A parameters does not yield a satisfactory fit to the scenario interference.

The fundamental reason that the Class A parameters that fit scenario interference cannot be calculated directly is that the canonical shape of the Class A pdf is strikingly different from the shape of the pdf that naturally arises from the scenario in Section 4.2. The pdfs of interference, given that exactly one interferer is transmitting, will be used to demonstrate the difference. This will be sufficiently accurate, because, as Figure 13 shows, interference from more than one interferer at a time has only a second-order effect on the pdfs for A=U=0.25.

The pdf of interference from one interferer for the scenario is proportional to $f_v(v)$ in (A-4) and is plotted as curve B in Figure A-1 for the scenario value E(v) = 1.

The corresponding canonical form for Class A noise for m=1 is $g\ e^{-gv}$, where $g \approx 1/E(v)$ when $\Gamma \ll 1$, as it is in this case. This function is plotted as Curve A in Figure A-1. Notice that the shape of Curve B is unlike that of Curve A.

The difference in shapes is even more dramatic when we convert the variable v from power to dBW. Figure A-2 shows the pdfs after this transformation. Again, Curve B is that for the scenario and Curve A comes from the canonical form of Class A interference. Curve A has the familiar shape of the nongaussian interference bump (see, for example, Figure 14 of the report).

However, the shape of Curve B is somewhat different from that of the scenario interference in Figure 14. This is because the scenario leading to Figure 14 included location variability in the propagation and had a statistical distribution of transmitter power, but the scenario in this Appendix did not. If there is no propagation variability, an interferer at the maximum range of 10 km produces exactly $1/(10^4)$ W, or -40 dBW interference. There are no interferers at greater ranges, so there is zero probability of lower levels of interference, and the pdf (Curve B of Figure A-2) has an abrupt jump at -40 dBW.

In the scenario leading to Figure 14, the power received from a fixed distance (like 10 km) has a statistical distribution, so that an interferer at a distance of 10 km might produce less than -40 dBW interference. Therefore there is no abrupt jump in the scenario interference in Figure 14--the left end of the interference bump has the same smooth shape as the distribution of transmission loss. If there had been no variability in transmission loss in the scenario in Section 4.2, the disparity between scenario noise and Class A noise in Figure 14 would have been even greater.

The disparity in the shapes of Curves A and B in Figure A-2 is the reason that direct calculation of Class A parameters from the scenario results in a worse fit than the approximate method described in Section 4.2 of the report.

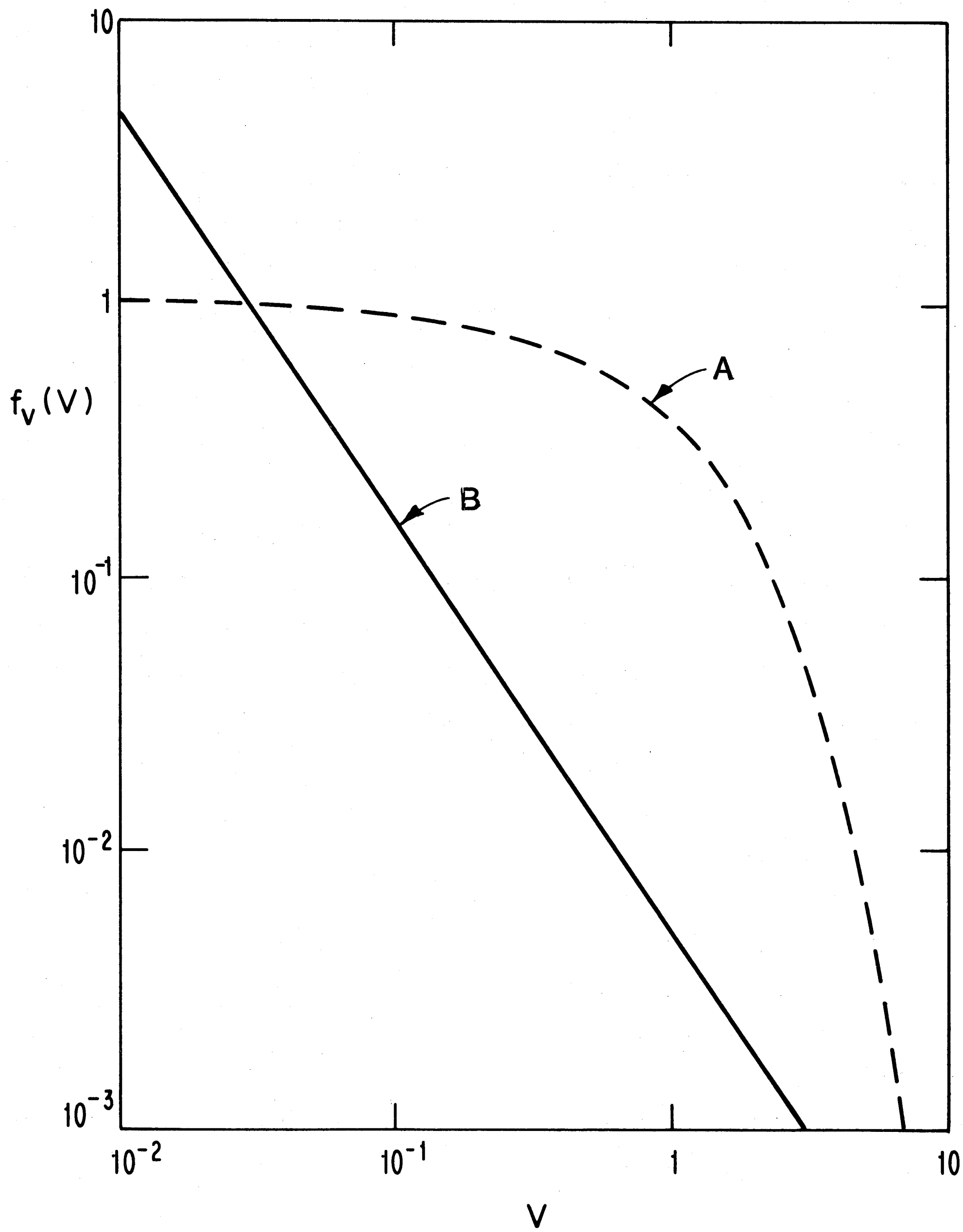

Figure A-1. The pdf of interfering power, in watts, from exactly one simultaneous interferer. Curve B is the pdf computed for the scenario; Curve A is the canonical form of Class A noise. Notice that the curves are on a log-log scale.

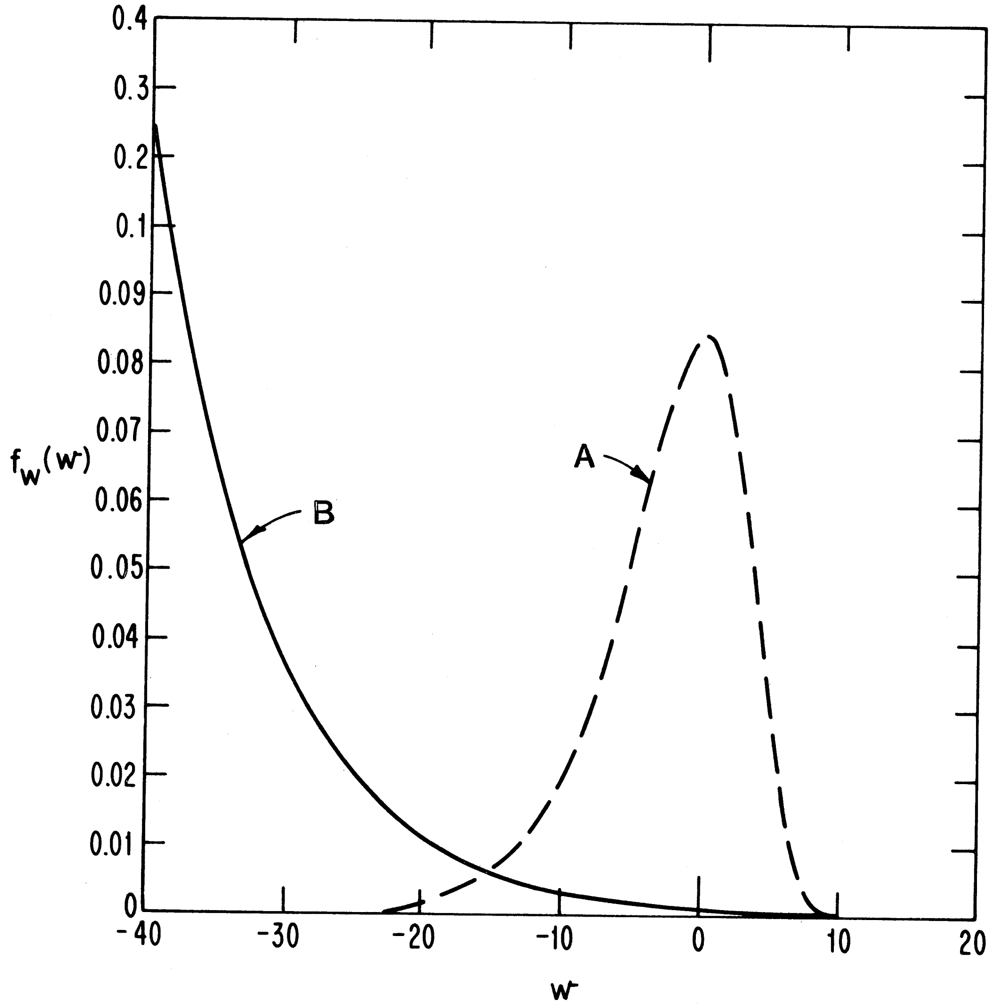

Figure A-2. The pdf of interfering power, in dBW, from exactly one simultaneous interferer. Curve B is the pdf computed for the scenario; Curve A is the canonical form of Class A noise.

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15. ABSTRACT *(A 200-word or less factual summary of most significant information. If document includes a significant bibliography or literature survey, mention it here.)*

Middleton's analytical formula for the probability of communications in a land mobile environment and Berry's computer program for probabilistic calculations of interference are shown to produce identical answers for identical inputs. The assumptions, input, and output of the two models are compared. Middleton's assumption that interfering sources transmit randomly and independently is of particular interest for mobile radio services. The effects of the greater variety of input assumptions possible with the computer method are illustrated numerically.

To use Middleton's formula to compute the probability of interference for a detailed scenario, the parameters of Class A noise must be determined from the scenario descriptors. Methods for finding these parameters are explored.

To illustrate possible applications, the number of statistically identical links that can operate with specified reliability in a given area is computed with the numerical model.

16. Key works *(Alphabetical order, separated by semicolons)*

Class A noise; communications probability; mobile radio; probabilistic EMC; probability of interference

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