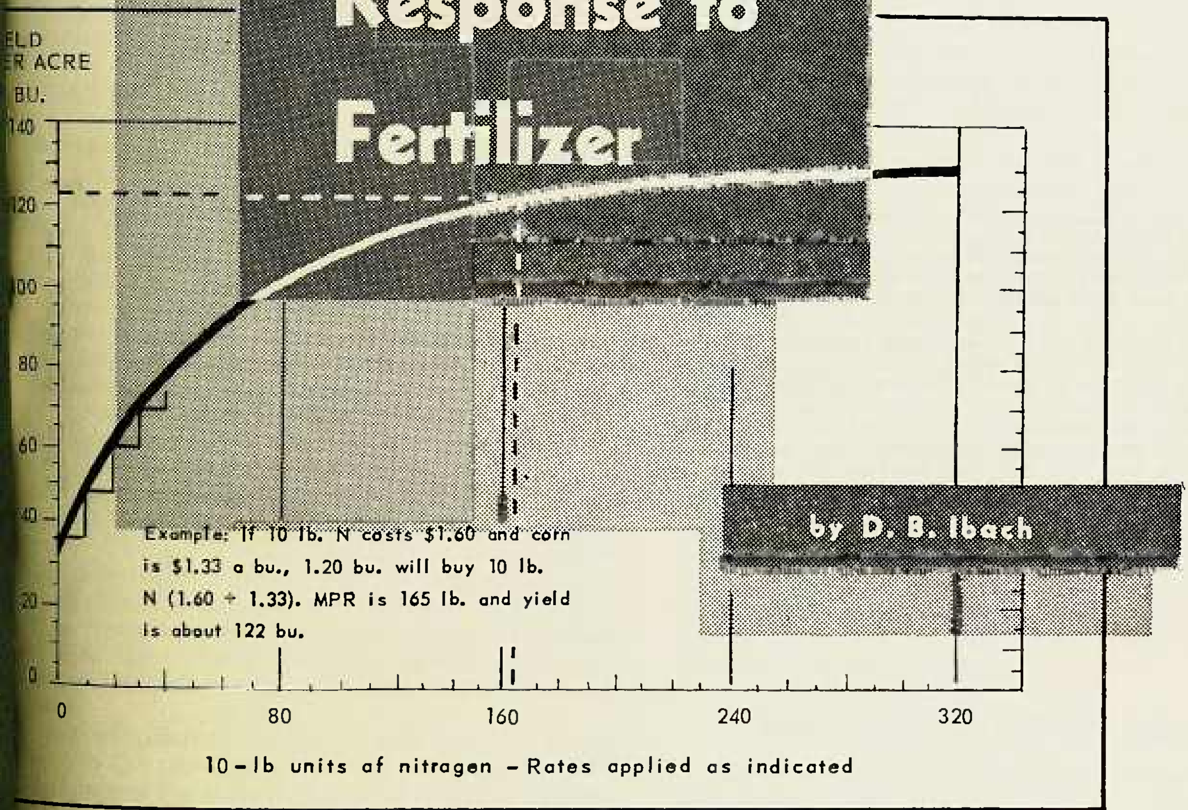
A Graphic Method of Interpreting Response to Fertilizer

by D. B. Ibach



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PREFACE

Recent rate experiments have provided an improved basis for exploring the nature of response to fertilizers. More rates of each of the principal plant nutrients—nitrogen, phosphoric oxide, and potash—are now applied at different levels of other yield-influencing factors than was done in the past. As results from more of these experiments become available, studies can be made as to the relative merits of different yield equations. Equations that are found to be satisfactory may be used in estimating yield response and returns per acre at different combinations of yield-influencing factors. These studies involve use of statistical methods, so that statements may be made with confidence for each important conclusion.

There is always need for continuing tests of adequacy of different yield equations and for determining confidence limits of estimates developed from the tests. Statistically trained researchers will need to make these evaluations. However, many research workers, teachers, and others will find graphic methods useful in obtaining the practical answers desired. Sums of squared residuals can then be used to indicate the amount of variability in the reported data. The method presented here is intended for those who have neither the time nor the facilities to obtain results that are statistically precise, but who are called upon to make recommendations regarding profitable use of fertilizer.

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SUMMARY

The exponential equation used here lends itself well to graphic methods of estimating response to fertilizers. The most profitable rate of a fertilizer can be read directly from a graphically fitted curve. The most profitable combination of two or more nutrients may be calculated readily after the constants of the equation are found by graphic methods for each nutrient. When they are carefully done, graphic methods yield results that closely approximate those obtained from mathematical determination of the constants. Graphic methods of using the exponential equation are easily followed by those unfamiliar with statistical techniques, who often need to make recommendations as to use of fertilizer based on results of rate experiments.

The important use of a yield equation in connection with fertilizer rate experiments is to predict the most profitable rates and combinations, and the yields of the crops at those rates. Possibilities of profitable substitution (economic) of one nutrient for another are limited. A decline in the price of the crop subsequent to application of fertilizer at a given rate is of little significance with respect to the optimum economic rate of application. The amount of reduction in return above cost of fertilizer because of failure to apply the most profitable rate depends on the shape of the curve of response to additional applications. Existence of risk factors will usually suggest applications somewhat below the calculated most profitable rates.

Use of different equations may produce results that fit the reported yields about equally well for that part of the surface that is of practical interest. Ease of handling the work and goodness of fit over the range within which applications would be most profitable for farmers with either limited or unlimited funds are important points to be considered in selecting an equation.

Experiments should include rates that encompass the total range in response, so that a reliable estimate of the maximum yield can be obtained. Also, an adequate number of rates in the area of greatest curvature is needed to give reliability to estimates of yield in that portion of the curve. If these conditions are met, recommendations based on fitted curves—graphic or otherwise—need not be questioned because of deviations on the steeper part of the curve or on the portion to the right of the most profitable rate. On the steep part of the curve, returns usually are so high that this part of the curve needs no special study even for the farmer with limited funds. Then, as the calculated most profitable rate is approached, the response to additional fertilizer is usually so small that risk factors outweigh possible gains from further increments in yield, even for farmers whose funds are unlimited.

The first pages of this handbook include discussion and illustration of a graphic method of fitting the curve, and of the results obtained. This part of the handbook, together with Appendix table 9, should be useful in developing recommendations and other illustrative material based on suitable response data. The rest of the handbook concerns the question of minimum cost combinations for a specified yield. It contains also a comparison of results from use of the exponential and the quadratic square-root equations.

THE YIELD EQUATION USED

The equation illustrated—an exponential form—was presented and discussed earlier.[2] It rests on the hypothesis that as fertilizer is added in units of uniform size, with other factors unchanged, increments in yield decrease at a constant per-

[1] This handbook supersedes U. S. Bur. Agr. Econ. F. M. 105, Determining Profitable Use of Fertilizers, 1953, by D. B. Ibach and S. W. Mendum. Acknowledgment is due S. W. Mendum and Burton L. French, Production Economics Research Branch, Agricultural Research Service, for helpful suggestions and assistance in the computations and for consultation on statistical methods, respectively.

[2] Ibach, D. B., and Mendum, S. W. DETERMINING PROFITABLE USE OF FERTILIZERS. U. S. Bur. Agr. Econ. F. M. 105, 70 pp., illus. 1953.

Spillman, W. J. USE OF THE EXPONENTIAL YIELD CURVE IN FERTILIZER EXPERIMENTS. U. S. Dept. Agr. Tech. Bul. 348, 66 pp., illus. 1933.

Spillman, W. J., and Lang, E. THE LAW OF DIMINISHING RETURNS. 178 pp., illus. New York and Chicago. 1924.

centage rate. This ratio between successive increments in yield may have a different numerical value, depending on growth conditions.[3] Under most field conditions, some factors usually operate to limit the yield that could be obtained if all conditions were ideal. For practical purposes, therefore, it is necessary that the method used allow for variation in the constants of the yield equation, according to conditions of the experiment.

For purposes of illustration here, the equation reflecting the hypothesis indicated above is written as $y=M\ (1-R^x)$. In this equation, y is the calculated yield at any specified level of the nutrient, M is the theoretical maximum yield attainable through use of fertilizer, for *conditions accompanying the reported yields*, R is the ratio of successive increments in yield—a constant having a value more than 0 but less than 1.0— and x is the quantity of the independent variable—fertilizer—changes in which are found to be associated with changes in yield. The term "$1-R^x$" is expressed as a decimal fraction, which represents the ratio between a particular yield and the maximum yield. The quantity of a plant nutrient (x) associated with yield refers to both soil content and applied portion. For example, when x refers to nitrogen (N), soil content is designated as n, and a is used to indicate the applied portion. Thus, $x=n+a$. Similarly, for phosphoric oxide (P_2O_5), $x=p+b$; and for potash (K_2O), $x=k+c$. The term "soil content" as used here is not meant in a strictly literal sense. The symbols n, p, or k designate constants that are functions of the yield equation used.

Results that involve one or more than one independent variable can be obtained. For combinations of independent variables, results are obtained at present only by fitting each regression curve at a specified level of each of the other nutrients. Thus far, no one has worked out for the exponential model a convenient method of solving the least-squares estimating equations when more than one input is applied at different levels of other inputs. Also, such a solution would not lend itself to simple graphic treatment. For the present, the multivariable form of this equation is used for any particular experimental data under the assumption that the rate (R) of response to a nutrient is the same at different specified levels of other nutrients. The results applied in this way to three factorial experiments are compared with those obtained from simultaneous estimates of values for another equation applied to the same data.

Graphic Use of the Exponential Equation

As the equation used is based on the assumption of a constant ratio (R) of decreasing increments in yield, a table of $1-R^x$ (yield ratio) values based on some specified value of R can be prepared. As Spillman had prepared such a table in terms of $R=0.8$,[4] this value is adopted (see table 9, Appendix). This table of yield ratio values may be put in the form of a graph, or graphs, in which these values on the vertical scale are plotted against number of units of the size required to make $R=0.8$. Connecting the points plotted in this way results in a "standard yield curve." Plotting this on different horizontal scales (unit values) results in standard yield curves of different shapes. For convenience, a specified value of R (0.8) is used, but reflection of different rates of response in different situations is permitted by the fact that the scale which indicates the size of a unit is different for each standard curve. The number of units associated with a particular yield ratio value is the same on all curves.

Several such curves should be prepared so that use of some part of one of them will result in estimated constants that closely approximate those mathematically determined, if rates used in the experiment cover a range wide enough to characterize the response and, particularly, to locate the area of the maximum yield. If rates do not extend well beyond the steeper part of the curve, only estimates of yield falling within the range of rates applied can be considered reliable. However, this limitation is often less serious with respect to the exponential equation than with others. If the pattern of reported yields gives evidence that no further response would be obtained, reliability of the value of M read from the curve for use in the prediction equation will be greatly increased.

Standard yield curves that are clear and thin should be drawn to dimensions that permit use of overlays on which yields can be plotted on a scale that is large enough to permit accurate readings. Figure 1 illustrates part of a standard yield curve. The horizontal line across the top of figure 1 represents M, the maximum yield That is, at M the $1-R^x$, or yield ratio, value is 1.0. For permanent use, standard yield curves may be photographed on layers of opaque plastic material.

FITTING THE STANDARD CURVE TO REPORTED YIELDS

Reported yields can be plotted against rates of application on cross-section paper. By using the overlay process, it is possible to find a portion of one of the standard curves that appears to approximate a least-squares fit to the data. Figure 2 illustrates a set of data plotted on an overlay sheet. Figure 3 shows figure 2 overlaid on the standard curve of figure 1. Figure 3 is enlarged

[3] E. A. Mitscherlich in Germany developed the hypothesis that for a given size of unit of a specified nutrient the ratio between successive increments in yield is constant for all crops and soils, provided no other factors limit the yield. This view is also advanced in the United States, primarily by O. W. Willcox. *See*

MITSCHERLICH, E. A., DAS ERGEBNIS VON ÜBER 27,000 FELD-DUNGUNGSVERSUCHEN. Ztschr. f. Pflanzenernähr. Düngung u. Bodenk. 38: 22–35. 1947.

WILLCOX, O. W. ABC OF AGROBIOLOGY. 323 pp., illus. New York. 1937.

[4] See footnote 2, p. 1.

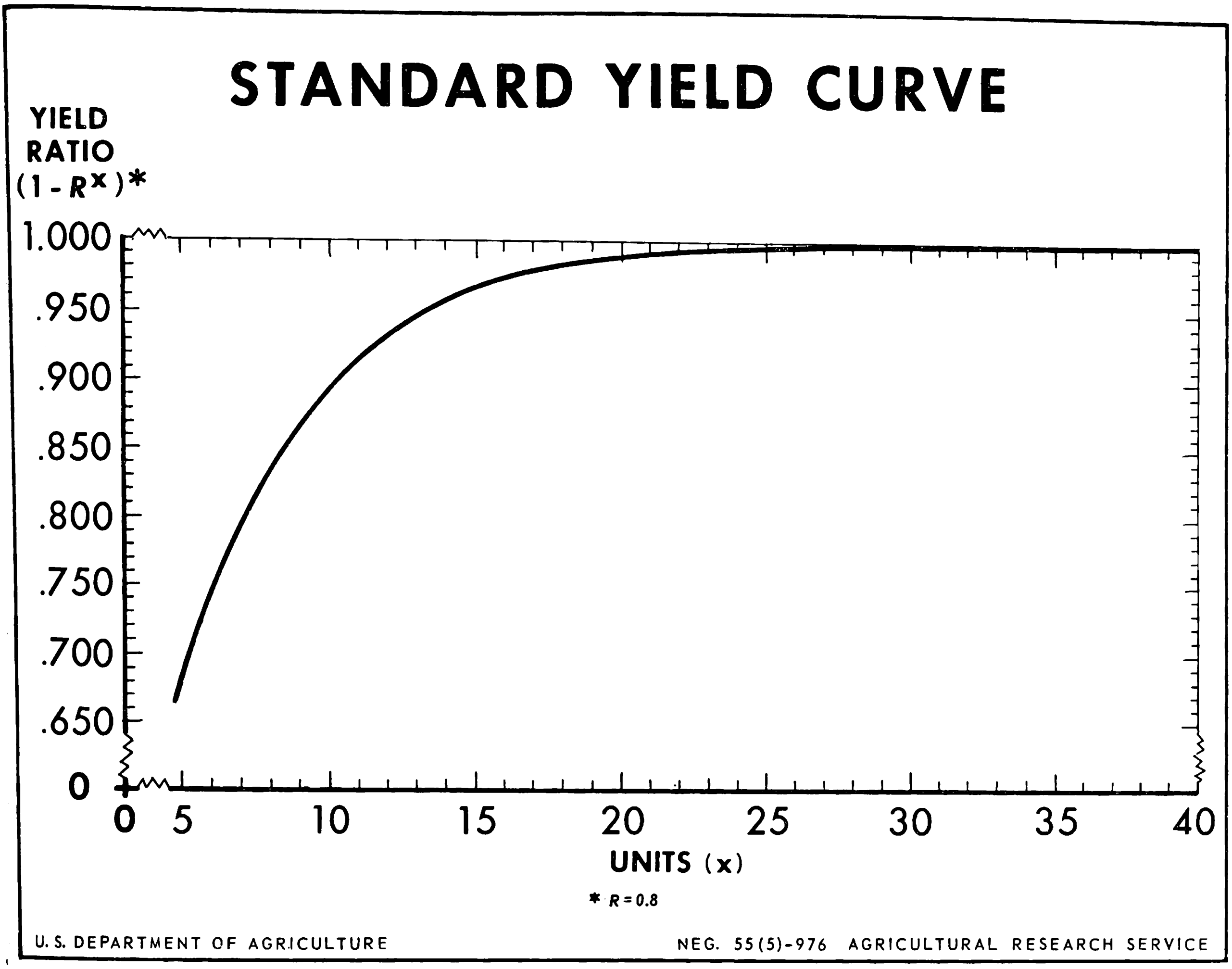

***Figure* 1.—Section of standard yield curve when R=0.8.**

to permit more exact estimates of fractional units and yield ratio values.

When a position on the standard curve is found so that the curve appears to fit the data, further refinement can be obtained by counting the deviations of reported yields both above and below the standard yield curve. These may be recorded and totaled in 2 columns, 1 for "+" and 1 for "—" deviations. When the sum of these 2 columns is as nearly 0 as can be obtained, the fit may be said to approximate the fit that could be determined mathematically, provided that the sum of squared deviations is smaller for the part of the curve tested than it would be for some other part of the curve or another standard yield curve. When the estimate of best fit is obtained, the maximum yield (M) is read at the point at which the yield scale on the overlay sheet intersects the top line, indicating that $1-R^x=1.0$ on the standard yield chart.

Thus, by repeated trials in which the position of the overlay sheet is shifted, it is possible by graphic means to obtain a fit to the curve that closely approximates one that is mathematically determined. Care must be taken to see that the overlay sheet is held in a position that "squares" with the grid marks at the edges of the sheet on which the standard yield curve is shown.[5]

RESULTS FROM BOTH GRAPHIC AND PRECISELY FITTED CURVES

Three experiments in 1952 and 1953, each carrying 12 rates of nitrogen on irrigated corn, were used in comparing results from use of the graphic method described with results obtained later when the least-squares fit was precisely determined. In determining the least-squares fit, the iterative method demonstrated by Stevens was carried out

[5] Even when an accurately computed curve is desired, this graphic approach saves time, in that often only one series of computations is required to find the true constants. Experience thus far indicates that use of this combination of the graphic approach plus one operation of the mathematical solution results in standard errors that are little improved by further iterations. Thus, for most practical purposes, the graphic device as a preliminary step makes use of the exponential function as easy and as rapid in developing a least-squares fit as does use of other functions that do not require iterative techniques.

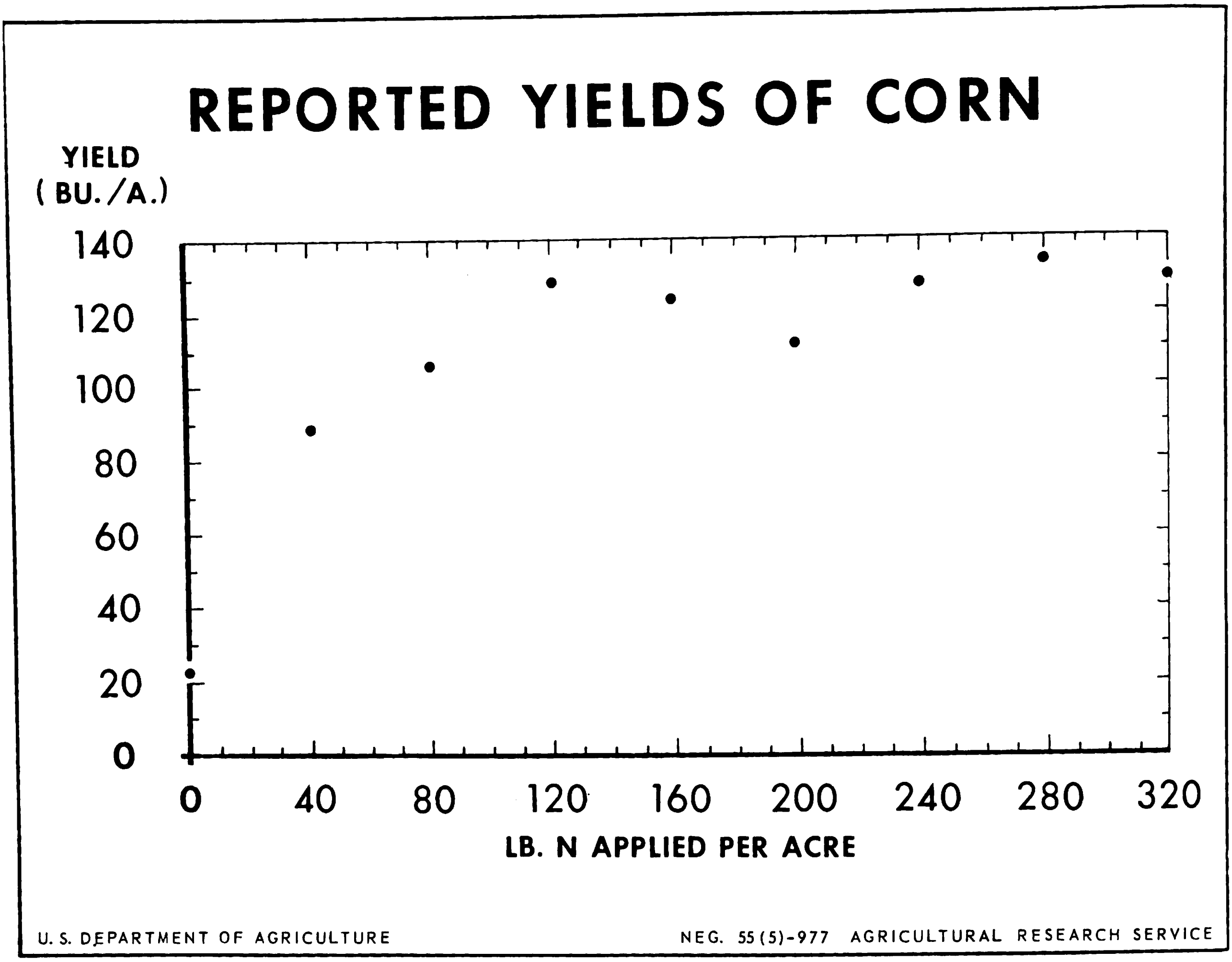

Figure 2.—Reported yields per acre of corn when nitrogen is the variable nutrient.

until no improvement in the fit could be obtained.[6] Results from the 2 methods are compared in table 1. Comparisons were made of all 12 rates of nitrogen and also for a smaller number of rates well distributed over the range. Results were about the same; they indicated that, in these instances, 5 or 6 rates distributed over the range were adequate for purposes of economic interpretation. In the 3 experiments, rates were carried to levels sufficiently high to characterize the response. Inclusion of such levels is important in obtaining reliable answers through use of any yield equation.

In 3 of the 6 comparisons between graphic and precisely determined curves, differences in yields at the most profitable rates were smaller than 1 standard error in the yield at the most profitable rates on the mathematically fitted curves. (Compare differences between table 1, cols. 3 and 4, with the standard errors shown in col. 5.) This means that these differences in yields are no greater than might be expected as a result of chance.

[6] STEVENS, W. L. ASYMPTOTIC REGRESSION. Biometrics 7: 247–267. 1951.

The effect of differences in the most profitable rates of nitrogen, as calculated from the two methods, is brought out in columns 8 to 10, table 1. In column 8 is shown the return above cost of fertilizer when it is calculated from the graphically fitted curve. Column 9 shows the same for the mathematically fitted curve. Then, using constants of this curve, the return above cost of fertilizer was calculated for the rate indicated as most profitable on the graphically fitted curve. These results are shown in column 10. Differences within these three columns are negligible. Differences between columns 9 and 10 represent what a farmer would lose per acre if he applied the graphically estimated most profitable rates instead of those calculated with precision. This assumes that conditions of the experiment were duplicated exactly in the field. The probability of this occurrence is remote.

Sums of squared residuals for the two curves are also shown in table 1. These sums are approximately twice as great for the graphic as for the mathematically fitted curve. Obviously, if the problem were one in which precision, rather than

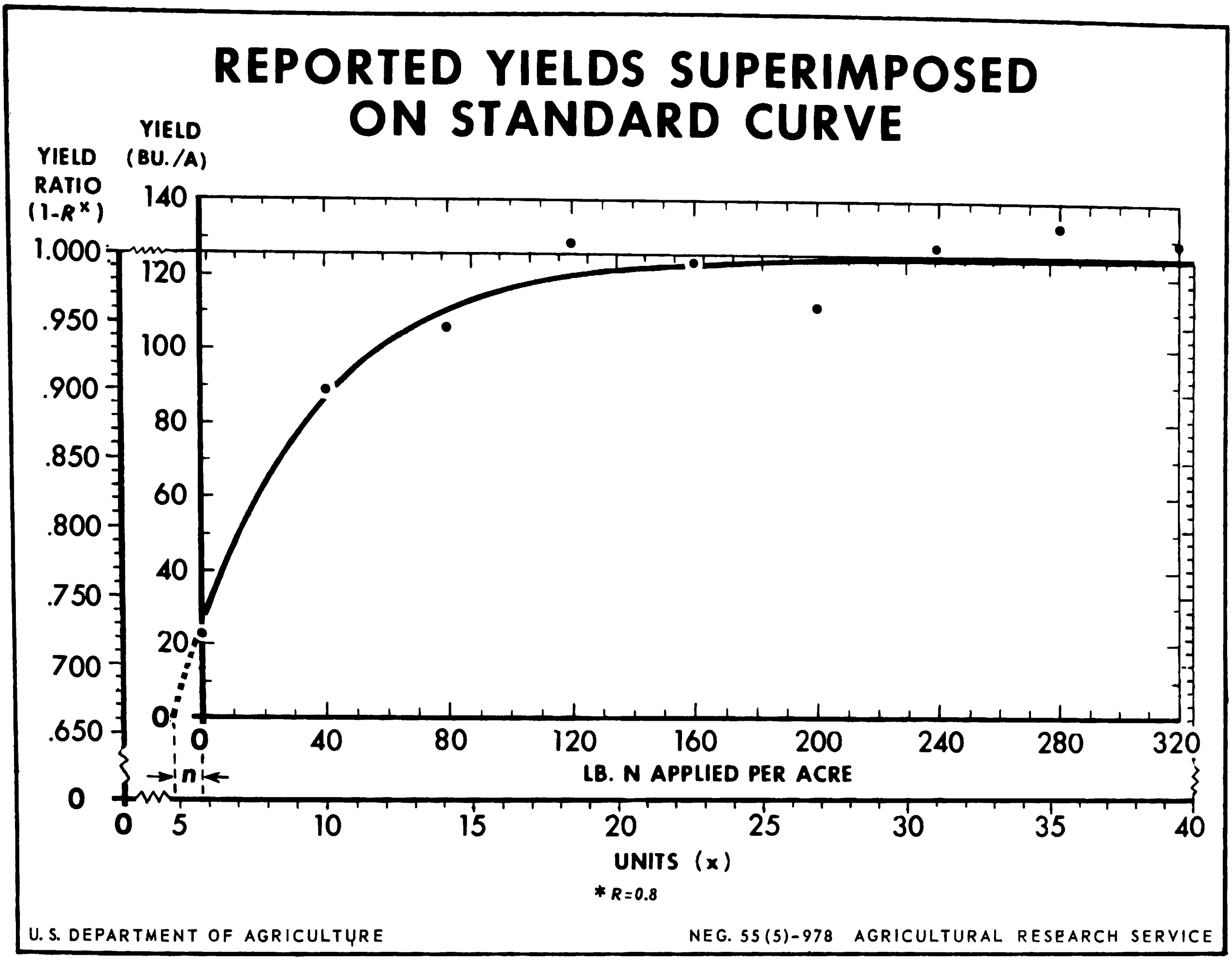

Figure 3.—**Reported yields of corn at different rates of nitrogen superimposed on standard yield curve.**

a basis for approximate field recommendations was needed, the mathematically fitted curve would be necessary. But in interpreting results for application in the field, approximations that can be made by the method described produce results so close to those obtained by the most accurate methods that the additional time required by the mathematically fitted curve is not justified. If the exponential equation is suited to the data at hand, the graphic method of fitting the curve is a useful device for those who need reliable answers quickly.

TABLE 1.—*Comparison of most profitable rates of nitrogen for corn, yields, residual sums of squares, and return per acre, as calculated from graphic approximation to least squares and from mathematically determined least-squares fits to the reported data, 1952–53* [1]

Experiment and series of data used	Most profitable rate of nitrogen—		Yield per acre at most profitable rate—		Standard error of yield at most profitable rate on fitted curve	Residual sums of squares—		Return per acre above cost of fertilizer—		
								At most profitable rate—		On fitted curve at most profitable rate on graphic curve
	On graphic curve	On fitted curve	On graphic curve	On fitted curve		On graphic curve	On fitted curve	On graphic curve	On fitted curve	
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
	Pounds	*Pounds*	*Bushels*	*Bushels*	*Bushels*			*Dollars*	*Dollars*	*Dollars*
Experiment 1, Hardy, Nebr.:										
6 rates spaced over range	136	153	119.11	119.90	±1.38	76.36	27.31	147.25	145.78	145.09
All 12 rates	134	164	118.99	121.63	±1.37	331.35	161.74	147.53	146.47	145.50
Experiment 2, Ontario, Oreg.:										
6 rates spaced over range	168	151	140.69	139.01	±5.03	85.40	74.96	173.23	173.23	173.08
All 12 rates	168	164	142.01	141.75	±1.71	270.29	254.78	174.65	175.04	175.07
Experiment 3, Prosser, Wash.:										
5 rates spaced over range	233	267	137.86	140.92	±2.71	121.27	52.80	158.52	157.39	156.45
All 12 rates	231	269	135.34	140.25	±1.99	519.96	307.20	155.24	156.12	154.98

[1] Based on data contained in U. S. Dept. Agr. Tech. Bul. 1141, A Method of Economic Analysis Applied to Nitrogen Fertilizer Rate Experiments on Irrigated Corn, by J. L. Paschal and B. L. French. [In press.]

GRAPHIC ESTIMATE OF YIELDS AND MOST PROFITABLE RATE WITH ONE NUTRIENT VARIED

An estimate of yield at any rate of application is read from the standard curve traced on the overlay sheet, which is scaled to show yields and rates of application. When only 1 independent variable is involved, it is not necessary to find constants of the yield equation. The yields are read directly from the curve, as indicated in figure 3. In this illustration, the curve is based on 9 rates of N, including no application, with no other fertilizer variable.

The most profitable rate of application is located by a simple calculation. Figure 4 represents the overlay portion of figure 3, with the curve traced in. The abscissa of figure 4 is scaled to indicate 10-pound units of application so that increments in yield associated with each 10-pound unit may be estimated. Suppose 10 pounds of N costs $1.60, including cost of application, and that a bushel of corn standing in the field before harvest is worth $1.40. Then 1.14 bushels of corn are equal in value to 10 pounds of N ($1.60/$1.40). This would mean that it would be profitable to apply N until the last 10 pounds added would result in an additional yield of 1.14 bushels of corn. So the most profitable rate is within the 13th 10-pound unit applied, because this unit is associated with the last increment in yield that is equal to or exceeds 1.14 bushels. Thus, the most profitable rate is between 120 and 130 pounds of N, and the yield at that rate is between 120 and 122 bushels.

If the standard yield curve and the overlay grid are plotted on a sufficiently large scale, these estimates may be made easily with sufficient accuracy for any practical purpose. For example, based on an accurately calculated least-squares fit, the most profitable rate of N was found to be 129 pounds, and the calculated yield at that rate is 121.6 bushels.

The particular shape of the curve in the area of greatest curvature may affect substantially the estimate of the most profitable rate. This serves to emphasize the importance of an adequate number of rates in that area, as well as in the area of the maximum yield.

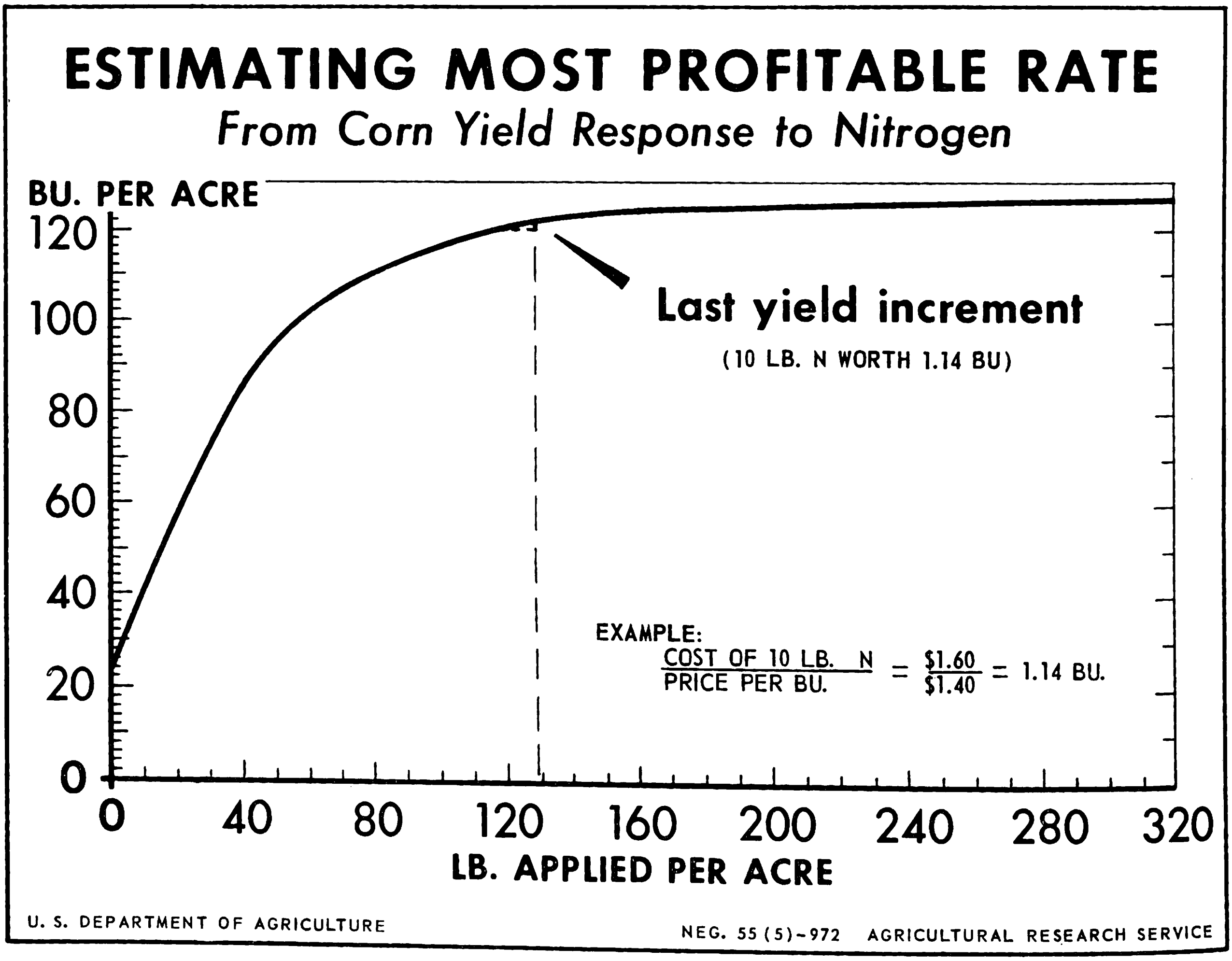

Figure 4.—Estimating the most profitable rate from corn yield response to nitrogen.

GRAPHIC ESTIMATE OF YIELDS WITH TWO NUTRIENTS VARIED

The illustration presented here is one in which the reported yields at different rates of N, used in developing the curve, were obtained when the application of P_2O_5 was held constant at 160 pounds per acre. Similarly, yields used in finding constants for the P_2O_5 curve were obtained when the application of N was held constant at 160 pounds. Graphic methods may be used in finding the constants of the curve for each nutrient.

In figure 3 the abscissa of the overlay (fig. 2) was purposely coordinated with the unit scale of the standard curve of figure 1. This was done to show the relationship between the scale in pounds for the nutrient applied and the number and size of units (x) when $R=0.8$ for the curve as fitted. Coordination of the overlay with the standard curve in this way permitted direct reading of n from the abscissa of the standard curve. It also permitted use of the scales on the standard curve for obtaining values from which is computed the size of a unit of N needed to make $R=0.8$, referred to as u_a. The reason for this careful coordination was to illustrate more closely the relationship of these constants to the curve.

However, in practice it is neither necessary nor feasible to obtain the precise coordination of scales on the overlay with those on the standard curve. All that is required is that one of the standard curves (many should be prepared) be found that fits the data, and that it be used only to obtain a reading of M on the ordinate of the overlay and of the desired estimated yields on the curve that are coordinate with rates of application designated on the abscissa of the overlay. The constants n and u_a are derived from these readings.

The general method suggested for estimating the values of the constants n and u_a (or p and u_b or k and u_c) involves use of two yields read from the fitted curve, each subject to the following restrictions applicable to some data. Near 0 application with low-level fertility, the curve rises so steeply that it is difficult to locate the point on that part of the curve that is coordinate with a specified rate of application designated on the abscissa of the overlay. Similarly, as the curve approaches the maximum yield (M), there is a wide range in rates of application indicated on the abscissa within which it is difficult to determine the rate that is coordinate with a specified yield on the curve. The problem is to avoid these extremes in order to obtain readings on the vertical and horizontal scales that are coordinate with each other, when the readings are to be used in estimating constants of the yield equation.

Accurate readings are more easily obtained when these extremes are avoided. Then, accurate calculations made from these readings will result in constants from which any yield calculated will fall on the standard curve from which the two yield readings are taken. A good general rule is to take the two yield readings at points that encompass the area of greatest curvature. Accuracy of the constants can be verified by using them to calculate a few yields over the range of observations. Lack of precision in reading yields from the fitted curve, or in the computations for finding constants, will have the greatest effect on accuracy of calculated yields at points that are farthest removed from the two readings taken.

Constants and Yields on the N Curve.—One of the constants on the N curve, M, has been estimated graphically as about 126.4 bushels (fig. 3). Next is needed the unit value, in terms of $R=0.8$, of the two components of x—soil content and applied portion. As indicated earlier, x for N is $n+a$. When the unit value of $n+a$ has been found for any rate of application, its associated yield ratio $(1-R^x)$ value multiplied by M results in the calculated yield at that rate.

Referring to figure 3, n (in pounds) is found by measuring the distance to the left from 0 application on the overlay sheet to the curve. In units, n is represented by the distance on the unit scale, x, between the point directly below zero application and the point below the intersection of the curve with the abscissa of the overlay sheet. The unit value coordinate with 0 application is 5.75. The curve crosses the abscissa of the standard yield chart at a little more than 4.8 units, estimated at 4.85. Therefore, the graphic estimate of n is 0.90 unit $(5.75-4.85)$.

However, as described earlier, n is more easily determined without requiring that the overlay be tailored to fit the dimensions of the standard curve. The yield at 0 application (y_o) is read from the curve as 23.3 bushels. At this yield, the yield ratio is 0.184 ($23.3\div M$, or 126.4). From a table of $1-R^x$ values (table 9), the x value of 0.184 is read as 0.91 unit. It differs only slightly from that read directly from the standard yield chart. This method does not require that fractional units be indicated on the standard chart. Nor does it require that the abscissa on the overlay sheet be coordinate with 0 yield.

Next is found the number of pounds of a (applied N) required to make $R=0.8$, which is referred to as u_a. Zero application is coordinate with 5.75 units, and the 320-pound application is coordinate with a little more than 39.5 units, estimated at 39.55. The difference—33.80 units—covers a range of 320 pounds. Thus, $320\div33.80=9.47$ pounds$=u_a$. (This value may be estimated in this way from any 2 points on the curve.) Then, pounds applied divided by u_a equals units applied. For example, when a is 160 pounds, a in units$=160\div9.47=16.89$ units. Thus, $n+a=0.90+16.89=17.79$ units, when the application is 160 pounds per acre.

However, as mentioned earlier, u_a, as well as n, is more easily determined in a way that does not require that the overlay be tailored to fit the standard curve. The following calculations illustrate the more practical use of the graphic method for this purpose.

$y_{160} \div M = 124.0 \div 126.4 = 0.981$ whose x value is 17.76
$y_o \div M = 23.3 \div 126.4 = 0.184$ whose x value is 0.91

Difference 16.85

Then $u_a = 160 \div 16.85 = 9.50$ pounds, close to the value read from the standard yield chart. As in the optional method of calculating n, x values are read from a table of $1-R^x$ values. The optional method may be faster if both a calculating machine and a table of $1-R^x$ values are available. The latter is needed in preparing the standard yield curves. However, if the optional method is used, the standard curves may be drawn to a smaller scale. When estimates n and u_a are to be read from the graph, the scale must be large enough to permit reading of fractional units, if results are to correspond closely with those obtained with more precise methods.

The yield at 160 pounds of a may be read directly from the curve as about 124 bushels. It is necessary to use constants of the N curve, together with those of other nutrient curves, in calculating yields at different combinations of nutrients. To illustrate use of the constants in calculating the yield at 160 pounds of a, it is recalled that the unit value of $n+a$ at 160 pounds of a is 17.79, based on direct graphic estimates. Reading upward from 17.79 on the unit scale, to the curve and across to the left to the yield ratio scale, the yield ratio is found to be about 0.981. Then, $y=M(1-R^x)=126.4\ (0.981)=124.0$ bushels. The yield at any point on the curve is calculated in the same way. Or, using the optional method, the value 0.981 may be read from a table of $1-R^x$ values.

Constants and Yields, the P_2O_5 Curve.—These are developed in the same way as illustrated for the N curve. Constants are based on response to 9 rates of P_2O_5, with applied N held at 160 pounds per acre. Read graphically, the constants are, M, 121.5 bu., p, 0.50 unit, and u_b, 7.63 pounds. At the 160-pound rate, b contains 20.97 units ($160 \div 7.63$). Thus, at this rate, $p+b$ is 21.47 units ($0.50+20.97$). Reading from figure 3, or from a table of $1-R^x$ values, the yield ratio value at 21.47 units is about 0.992. As $y=M(1-R^{p+b})$, $y=121.5\ (0.992)$, or 120.5 bushels. As the method is the same as that for the N curve, a graphic illustration is not included.

Maximum Yield for Any Combination of Nutrients.—The combination, 160 pounds of N and 160 pounds of P_2O_5, is found to occur on both curves. But in estimating a least-squares fit of each curve, the yields at this combination are not quite the same. Both curves are taken into account in estimating the maximum for the combination, subsequently referred to as M_2.

On the N curve, the yield at 160 pounds of N and 160 pounds of P_2O_5 is 124.0 bushels. Units of $n+a$ are 17.79 and the corresponding yield ratio value is 0.981. At this combination on the P_2O_5 curve the yield is 120.5 bushels. Units of $p+b$ are 21.47 and the corresponding yield ratio value is 0.992. The product of these 2 yield ratios is 0.973, or 0.981×0.992. Thus M_2, as calculated from the N curve, is $124 \div 0.973$, or 127.4 bushels. On the P_2O_5 curve, M_2 is calculated as $120.5 \div 0.973$, or 123.8 bushels. The accepted value of M_2 for purposes of subsequent calculations is 125.6, or the average of the values calculated from the 2 curves. In calculating yields at any combination, the process is reversed, and $y=M_2(1-R^{n+a})(1-R^{p+b})$.

Inclusion of a third nutrient, K_2O, involves the same steps. When the graphic approximation to least squares has been found, the maximum (M) is read from the standard yield chart. As for N and P_2O_5, x consists of 2 parts for K_2O, referred to as k and c, respectively. Then, y on the K_2O curve is calculated as $y=M(1-R^{k+c})$. The yield on the K_2O curve is estimated for the combination that is common to the 3 curves. Then, M_3 is calculated from that point on each yield curve as:

$$M_3 = y/(1-R^{n+a})(1-R^{p+b})(1-R^{k+c})$$

and the average of the 3 values of M_3, thus obtained, is used in all subsequent calculations that involve combinations of N, P_2O_5, and K_2O.

Yields at Any Combination

Using as M_2 the value 125.6, the yield at any combination of N and P_2O_5 is calculated as $y=M(1-R^{n+a})(1-R^{p+b})$. The value of n has been established at 0.90 unit and the value of p at 0.50 unit. On the N curve, u_a was found to be 9.47 pounds and on the P_2O_5 curve u_b was found to be 7.63 pounds. Thus, a and b in units are derived as pounds of applied N (a) and P_2O_5 (b) divided by 9.47 and 7.63, respectively. Thus, 160 pounds of applied N is equivalent to 16.89 units so that $n+a$ at 160 pounds of a is 17.79 units, whose $1-R^x$ value is 0.981. Similarly, using values of p and u_b as found on the P_2O_5 curve, $1-R^{p+b}$ at 160 pounds b is 0.992. Thus, the calculated yield at 160 pounds N and 160 pounds P_2O_5 is $y=M_2(1-R^{n+a})(1-R^{p+b})= 125.6(0.981)(0.992)=125.6(0.973)=122.2$ bushels. Yields for any combination are calculated in the same way.

The calculated yields in table 2, which are based on mathematically determined constants, are almost identical to those estimated when constants are determined by a graphic approximation to least squares, as illustrated earlier. Values of the constants are slightly different because of the small inaccuracies inherent in the graphic readings. Mathematical determination of the constants on which the calculated yields in table 2 are based were developed by the method suggested by Stevens.[7] For purposes of table 2, yields were calculated from accurately determined

[7] See footnote 6, p. 4. In his article, Stevens illustrates determination of standard errors of constants of the exponential equation.

constants so that standard errors could be computed.

The adequacy of the equation in describing yield response in this experiment is indicated by the multiple correlation coefficient of 0.9764. The standard errors of M and A (increase in yield attainable from use of fertilizer) on the N curve are 3.80 and 8.55 bushels, respectively, under the assumption that departures from the curve are normally distributed; s_R is 0.0746. On the P_2O_5 curve, s_M is 4.75 bushels, s_A is 11.69 bushels, and s_R is 0.1012. The standard error of estimate of calculated yields for the 57 treatments included in the experiment amounts to 10.43 bushels.

ECONOMIC COMBINATIONS OF NUTRIENTS

When 2 or more inputs are varied, each at different levels of the others, the estimated most profitable combination cannot be read simply from a standard yield curve, as may be done for only 1 variable. But with a calculating machine, economic combinations at specified nutrient costs, price of the crop required to render each combination most profitable, and other important determinations can be made easily. These calculations are based on values that have been determined by graphic methods. The work of computing a series of economic combinations and associated crop prices is reduced to a clerical job, if organized as successive steps to be performed by a calculating machine operator.

Table 3 presents a method of calculating most profitable combinations associated with different prices of the crop, gross returns and returns above cost of fertilizer, and reduction in returns from adding more or less fertilizer than would be most profitable. The method shown in table 3 also permits easy calculation of returns per dollar spent for fertilizer for each of a series of additional combinations of fertilizer applied. These returns, shown on the last line of table 3, are additional returns per dollar cost of the additional quantity of fertilizer indicated. At precisely the most profitable combination the added return per additional dollar is exactly \$1.00.

Item 1 in table 3 consists of a series of arbitrarily selected rates of one of the nutrients, in this case P_2O_5. At the indicated cost per pound of each nutrient, the rate of N that would be most profitable to use with any rate of P_2O_5 is derived. Following this is derived the price per unit of the crop required to render a particular combination most profitable.

Two additional values r' and r'' are introduced at this point. These are the costs of the quantities (u) of N and P_2O_5, respectively, when $R=0.8$. For example, with N at \$0.16 per pound, $r' = \$0.16 \times 9.47$ pounds, or \$1.52. Similarly, at \$0.10 a pound for P_2O_5, $r'' = \$0.10 \times 7.63$ pounds, or \$0.763. In setting up the calculations, the cost per pound of nutrient should include the estimated cost of application. The derived price of the crop should be interpreted as the price of the crop as it stands in the field before harvest. Each combination will then represent the one at which the farmer will realize maximum profit from fertilizer at the indicated crop price-fertilizer cost relationship. Figure 5 shows the changes in the most profitable combination over a wide range in price of the crop. Data for figure 5 were prepared as indicated in table 3, except that a wider range in rates was used than is shown in that table.

In table 3, the combination 129 pounds N and 130 pounds P_2O_5 is associated with a price of \$1.42 a bushel. This combination may then be regarded as the most profitable one at that price for conditions of the experiment.[8]

Effect of Applying More or Less Than Most Profitable Rates.—The effect of applying lower or higher rates than those most profitable for a specified price are indicated in figure 6, which is based on data from table 3. Here the price specified is \$1.42 a bushel, and the combination most profitable at that price is 129 pounds of N and 130 pounds of P_2O_5. But if the application is only 81 pounds N and 90 pounds P_2O_5, with the price for corn still \$1.42, the return would be \$9.42 less per acre than if the combination most profitable for that price had been applied. Different combinations are indicated in figure 6. One of these, 129 pounds of N and 130 pounds of P_2O_5, is

[8] See Appendix for method of calculating economic combinations of 3 nutrients, most profitable rate of 1 nutrient, and most profitable combination of 3 nutrients.

TABLE 2.—*Calculated yields per acre of corn for different combinations of N and P_2O_5 in a fertilizer rate experiment, Iowa, 1952* [1]

Pounds of P_2O_5 applied per acre	Pounds of N applied per acre								
	0	40	80	120	160	200	240	280	320
	Bushels	*Bushels*	*Bushels*	*Bushels*	*Bushels*	*Bushels*	*Bushels*	*Bushels*	*Bushels*
0	2.45	9.07	11.64	12.64	13.03	13.18	13.24	13.26	13.27
40	16.89	62.49	(80.25)	(87.12)	89.79	90.83	(91.23)	(91.38)	91.44
80	21.27	(78.69)	101.05	(109.70)	113.06	(114.37)	114.87	(115.07)	115.14
120	22.60	(83.60)	(107.36)	116.55	120.12	(121.51)	(122.04)	122.25	122.33
160	23.00	85.09	109.27	118.62	122.25	123.66	124.21	124.42	124.51
200	23.12	85.54	(109.84)	(119.25)	122.90	124.32	(124.87)	(125.08)	125.16
240	23.16	(85.67)	110.02	(119.44)	123.00	(124.51)	125.06	(125.28)	125.36
280	23.17	(85.71)	(110.07)	119.49	123.15	(124.58)	(125.13)	125.34	125.42
320	23.18	85.73	110.09	119.52	123.18	124.60	125.15	125.36	125.45

[1] Based on constants resulting from a mathematically fitted curve of response to N at the 160-pound level of P_2O_5, and of response to P_2O_5 at the 160-pound level of N. Yields shown in parentheses are at combinations not included in the experiment. See the following bulletin for reported yields of this experiment: HEADY, E. O., PESEK, J. T., and BROWN, W. G. CROP RESPONSE SURFACES AND ECONOMIC OPTIMA IN FERTILIZER USE. Iowa Agr. Expt. Sta. Res. Bul. 424, pp. 292–332, illus. 1955.

TABLE 3.—*Steps in calculating economic combinations of N and P_2O_5 for corn at specified costs per pound of the nutrients, price of the crop required to render each combination most profitable, yields and returns per acre at each combination* [1]

Item and description or derivation	Values						
	(1)	(2)	(3)	(4)	(5)	(6)	(7)
1. b, or applied P_2O_5 ..... pounds	90	100	110	120	130	140	150
2. b in units [item 1 divided by u_b; u_b equals 7.63] ..... units	11.80	13.11	14.42	15.73	17.04	18.35	19.66
3. $p+b$ [item 2 plus soil content (p); p equals 0.50] ..... do	12.30	13.61	14.92	16.23	17.54	18.85	20.16
4. $1-R^{p+b}$ [read from standard yield curve or from table 9] ..... decimals	.936	.952	.964	.973	.980	.985	.989
5. R^{p+b} [1.0 minus item 4] ..... do	.064	.048	.036	.027	.020	.015	.011
6. $(1-R^{p+b})$ (R^{p+b}) [item 4 times item 5] ..... do	.05990	.04570	.03470	.02627	.01960	.01477	.01088
7. r'' $(1-R^{p+b})$ [r'' equals cost per pound P_2O_5 (10 cents) times u_b (7.63 pounds), or $0.763] ..... numbers	.71417	.72638	.73553	.74240	.74774	.75156	.75461
8. r' (R^{p+b}) [r' equals cost per pound N (16 cents) times u_a (9.47 pounds), or $1.52] ..... numbers	.09728	.07296	.05472	.04104	.03040	.02280	.01672
9. r'' $(1-R^{p+b})+r'$ (R^{p+b}) [item 7 plus item 8] ..... do	.81145	.79934	.79025	.78344	.77814	.77436	.77133
10. $(1-R^{p+b})$ $(R^{p+b}) \div (r''$ $(1-R^{p+b})+r'$ $(R^{p+b}))$ [item 6 divided by item 9] ..... decimals	.07382	.05717	.04391	.03353	.02519	.01907	.01411
11. R^{n+a} [item 8 divided by item 9] ..... do	.11988	.09128	.06924	.05238	.03907	.02944	.02168
12. $1-R^{n+a}$ [1.0 minus item 11] ..... do	.88012	.90872	.93076	.94762	.96093	.97056	.97832
13. $n+a$ [read from standard yield curve or from table 9] ..... units	9.51	10.73	11.97	13.22	14.53	15.80	17.17
14. a [$n+a$ minus n, n equals 0.91] ..... do	8.60	9.82	11.06	12.31	13.62	14.89	16.26
15. Most profitable rate of N to apply [item 14 times u_a, or 9.47] ..... pounds	81	93	105	117	129	141	154
16. M_2 times item 10 [M_2, or theoretical maximum yield, equals 125.64 bushels] ..... numbers	9.2747	7.1828	5.5169	4.2127	3.1649	2.3960	1.7728
17. Item 16 times $-ln0.8$ [$-ln0.8$ equals 0.22314] ..... do	2.0696	1.6028	1.2310	.9400	.7062	.5346	.3956
18. v, or value per unit of crop in field [1.0 divided by item 17] ..... dollars	.48	.62	.81	1.06	1.42	1.87	2.53
19. $(1-R^{p+b})$ $(1-R^{n+a})$ [item 4 times item 12] ..... decimals	.824	.865	.897	.922	.942	.956	.968
20. y, or calculated yield [M_2 (or 125.64 bushels) times item 19] ..... bushels	103.53	108.68	112.70	115.84	118.35	120.11	121.62
21. Gross return at $1.42 per bushel [item 20 times $1.42] ..... dollars	147.01	154.33	160.03	164.49	168.06	170.56	172.70
22. Cost of fertilizer [item 2 times $0.763 plus item 14 times $1.520] ..... dollars	22.07	24.93	27.81	30.71	33.70	36.63	39.72
23. Return above cost of fertilizer [item 21 minus item 22] ..... do	124.94	129.40	132.22	133.78	134.36	133.93	132.98
24. Reduction in item 23 with corn at $1.42, at different rates ..... do	9.42	4.96	2.14	.58	.00	.43	1.38
25. Increase in gross return at additional combinations ..... do	----------	7.32	5.70	4.46	3.57	2.50	2.14
26. Cost of additional rates of fertilizer ..... do	----------	2.86	2.88	2.90	2.99	2.93	3.09
27. Changes in return per dollar cost of fertilizer, average each combination ..... dollars	----------	2.56	1.98	1.54	1.19	.85	.69

[1] Based on data from—HEADY, E. O., PESEK, J. T., and BROWN, W. G. CROP RESPONSE SURFACES AND ECONOMIC OPTIMA IN FERTILIZER USE. Iowa Agr. Expt. Sta. Res. Bul. 424, pp. 292–332, illus. 1955.

most profitable if corn is $1.42 a bushel. But, if 90 pounds of P_2O_5 are to be used, the best rate of N will be 81 pounds. Table 3 shows that the price at which this lower combination would be most profitable is only $0.48 a bushel.

Some verification of estimates in figure 6 may be had by comparing yields at the specified combinations with those reported at combinations most nearly similar. For example, the reported yields at 40 pounds of N and 40 pounds of P_2O_5 was 71 bushels, compared with 66 bushels calculated at 37 pounds of N and 50 pounds of P_2O_5. Also, at 80 pounds each of N and P_2O_5, the reported yield was 107 bushels compared with calculated yields of 97 bushels at 70 pounds of N and 80 pounds of P_2O_5, and 104 bushels at 81 pounds of N and 90 pounds of P_2O_5. Going back to still lower rates, the calculated yield of 36 bushels at 17 pounds of N and 30 pounds of P_2O_5 appears to be a reasonable estimate in view of the 28-bushel yield reported at 0 pounds of N and 40 pounds of P_2O_5.

Effect of Change in Price of Crop After Application.—The effect of a change in price of the crop from the price anticipated at the time of application is indicated in figure 7. Here again, the price expected is $1.42 a bushel, and the combination applied is the one most profitable at that price. Suppose the price falls to only $1.00 a bushel. How much are returns per acre reduced as compared with what they would have been if the combination most profitable for $1.00 corn had been applied? According to figure 7, the reduction in this instance would be only about $0.50 per acre. Figure 7 indicates that only an extreme drop in price would make any substantial difference in returns from application of what is expected to be the most profitable rate and application of the rate that would have been most profitable for the price actually received. The probability of reduction in returns from application of too much fertilizer because of a decline in price after application apparently is a relatively minor risk.

But even if the price received is $1.42 a bushel, there are other risks, and the farmer may not choose to apply the rates calculated as most profitable for that price. For example, the most profitable combination at that price (129 pounds N and 130 pounds P_2O_5) costs $5.89 more than the one that would be most profitable for only $0.81 corn—105 pounds N and 110 pounds P_2O_5 (table 3). If the crop is worth $1.42 at harvesttime, the return from application of fertilizer at the lower rates would be $2.14 per acre less than from application at the higher rates indicated—the ones most profitable at that price. But the natural hazard may be so great that the better choice would be to take the chance of losing the $2.14 if everything goes right than to lose the additional $5.89 if unfavorable weather or other damage were to reduce the yield to that associated with the lower rates, even though the higher rates are applied.

These are the kinds of odds from which the farmer must choose. The calculated most profitable rates provide starting points from which a

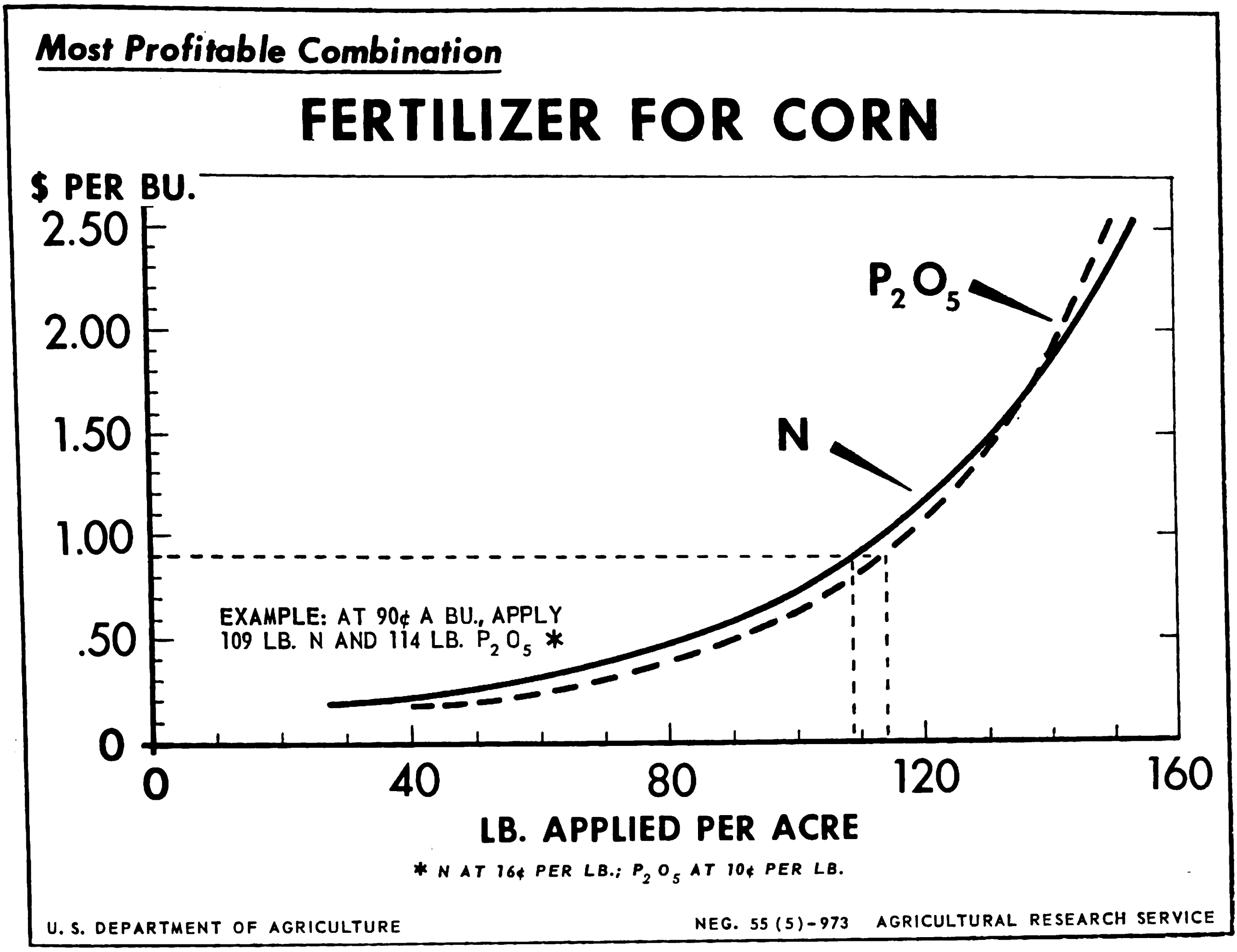

Figure 5.—Most profitable combination of N and P_2O_5 for corn at different prices per bushel.

farmer may estimate the rates he should apply, based on knowledge of the most probable environmental conditions. Some of the latter can be controlled by his own actions. Use of good cultural practices greatly increases the chances of realizing anticipated returns from the calculated most profitable rates.

MINIMUM COST NUTRIENT COMBINATIONS FOR A SPECIFIED YIELD

The idea that different combinations of nutrients may be applied to obtain a specified yield implies possibilities of substitution. Physiologically, one nutrient does not substitute for another in the plant. At least, a minimum quantity of each nutrient is required for healthy balanced growth. The plant may draw more heavily on the soil supply of a nutrient if the applied portion of that nutrient is inadequate to provide the physiological balance needed for the yield obtained. Thus, under certain nutrient-cost relationships, it may pay to "mine" the soil in the short run in order to obtain maximum returns. This might mean that later on the response to application of the depleted nutrient would become so great that, even though its cost per unit were still relatively high, more of it should be added to obtain a specified yield at minimum cost.

In deciding whether there are possibilities for profitable substitution between applied nutrients in obtaining a specified yield, it is necessary to know: (1) Different combinations of applied nutrients at which the specified yield would be obtained (equal product combinations) and (2) the relationship between the costs per unit of the nutrients that would be necessary in order to make each of these equal product combinations most profitable.

The answer to the second problem will indicate the marginal rate of substitution of one applied nutrient for another. Thus, the most profitable combination was calculated with N at $0.16 and P_2O_5 at $0.10 a pound. The ratio of the cost of N to the cost of P_2O_5 is 1.6 to 1.0. Therefore, in obtaining the yield at the most profitable combination for the conditions stated, 1 pound of applied P_2O_5 substitutes for 1.6 pounds of applied N; or 1 pound of applied N substitutes for 0.625 pound of applied P_2O_5 (1.0/1.6). If the nutrient cost ratio

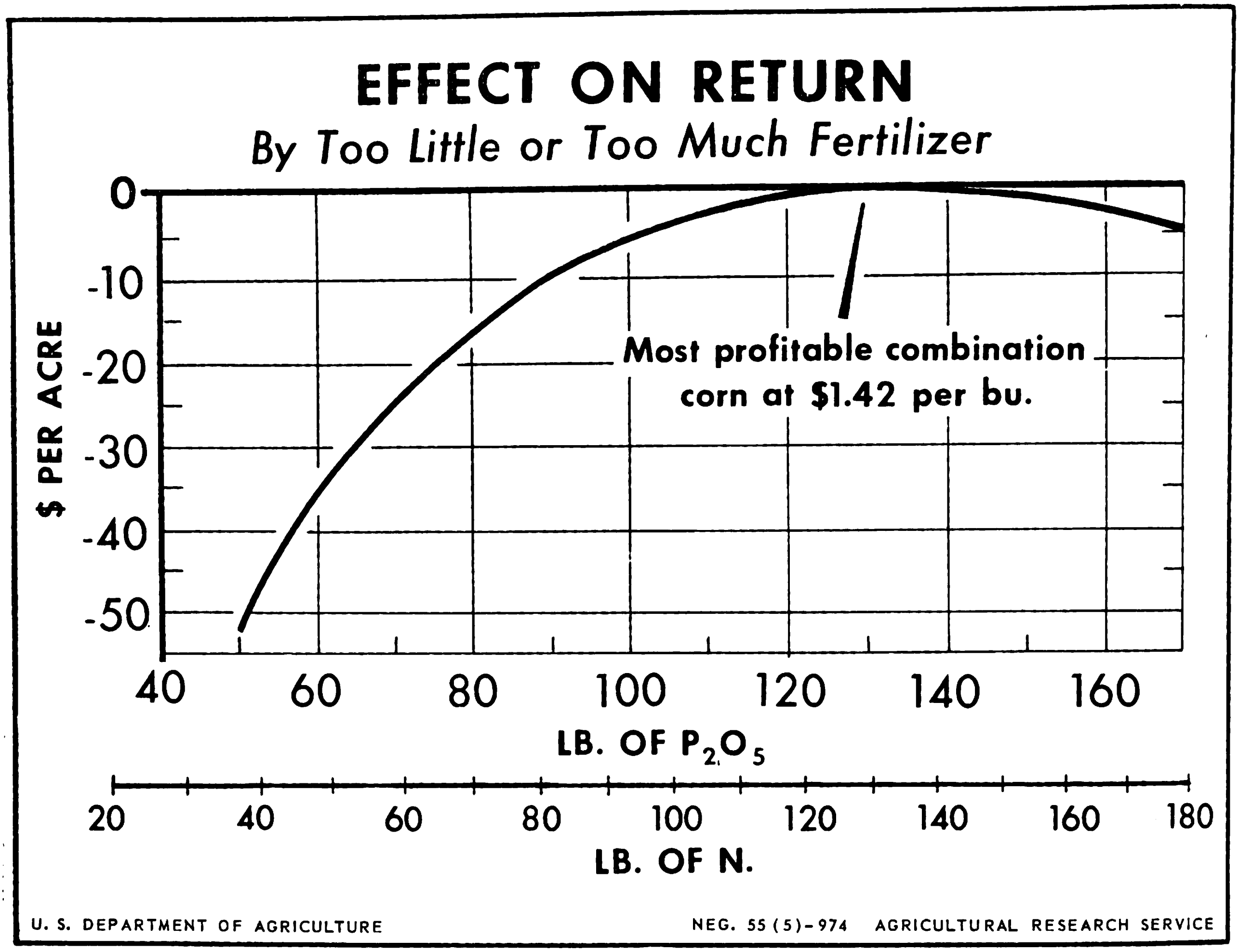

***Figure 6.*—Effect on return from corn by too little or too much fertilizer.**

should change substantially and if it is possible to obtain the specified yield at different combinations of nutrients, it would be worthwhile to know the marginal rates of substitution over a considerable range of such combinations. The relevant range would be just broad enough to include nutrient cost ratios that are at all probable. For example, it might be possible to obtain the specified yield at a combination that would be most profitable only if N cost 10 or more times as much per pound as P_2O_5. Obviously, such a cost relationship is not probable.

Determination of the marginal rate of substitution is illustrated here by referring to the yield at the most profitable combination for the estimated current price. The estimated current price is $1.42 and the yield at the combination most profitable at that price is 118.35 bushels (item 20, col. 5, table 3). The theoretical maximum yield (M_2) was found to be 125.64 bushels. Then $y \div M_2 = 118.35 \div 125.64 = 0.942$ (item 19, col. 5, table 3). Equal product combinations are determined as:

$$(y \div M_2) \div (1 - R^{n+a}) = 1 - R^{p+b}; \text{ or,}$$
$$(y \div M_2) \div (1 - R^{p+b}) = 1 - R^{n+a}$$

At the most profitable combination, $1-R^{n+a}$ is 0.96093 (item 12, col. 5, table 3). Then at the rate of b required for the specified yield, $1-R^{p+b}$ is $0.942 \div 0.96093$, or 0.980. Item 4, column 5, table 3, shows that this is the $1-R^x$ value associated with 130 pounds of P_2O_5, from which, at the stated nutrient costs, is calculated the rate of N (129 pounds) that would be most profitable to use with that rate of P_2O_5. When $1-R^{p+b}$ is 0.980, $p+b$ equals 17.54 units. As u_b is 7.63 pounds, b (applied P_2O_5) amounts to 130 pounds.

The discussion above illustrates calculation of equal product combinations of two independent variables, starting with a specified rate of one of them. The marginal rate of substitution (*MRS*) at each equal product combination is calculated as:

$$MRS \text{ in units} = \frac{R^{p+b}}{R^{n+a}} \cdot \frac{1-R^{n+a}}{1-R^{p+b}}$$

Substituting in the equation the values for the most profitable combination when corn is $1.42 (col. 5, table 3):

$$\frac{0.020}{0.039} \cdot \frac{0.961}{0.980} = \frac{0.01922}{0.03822} = 0.50288 \text{ unit of N, which}$$

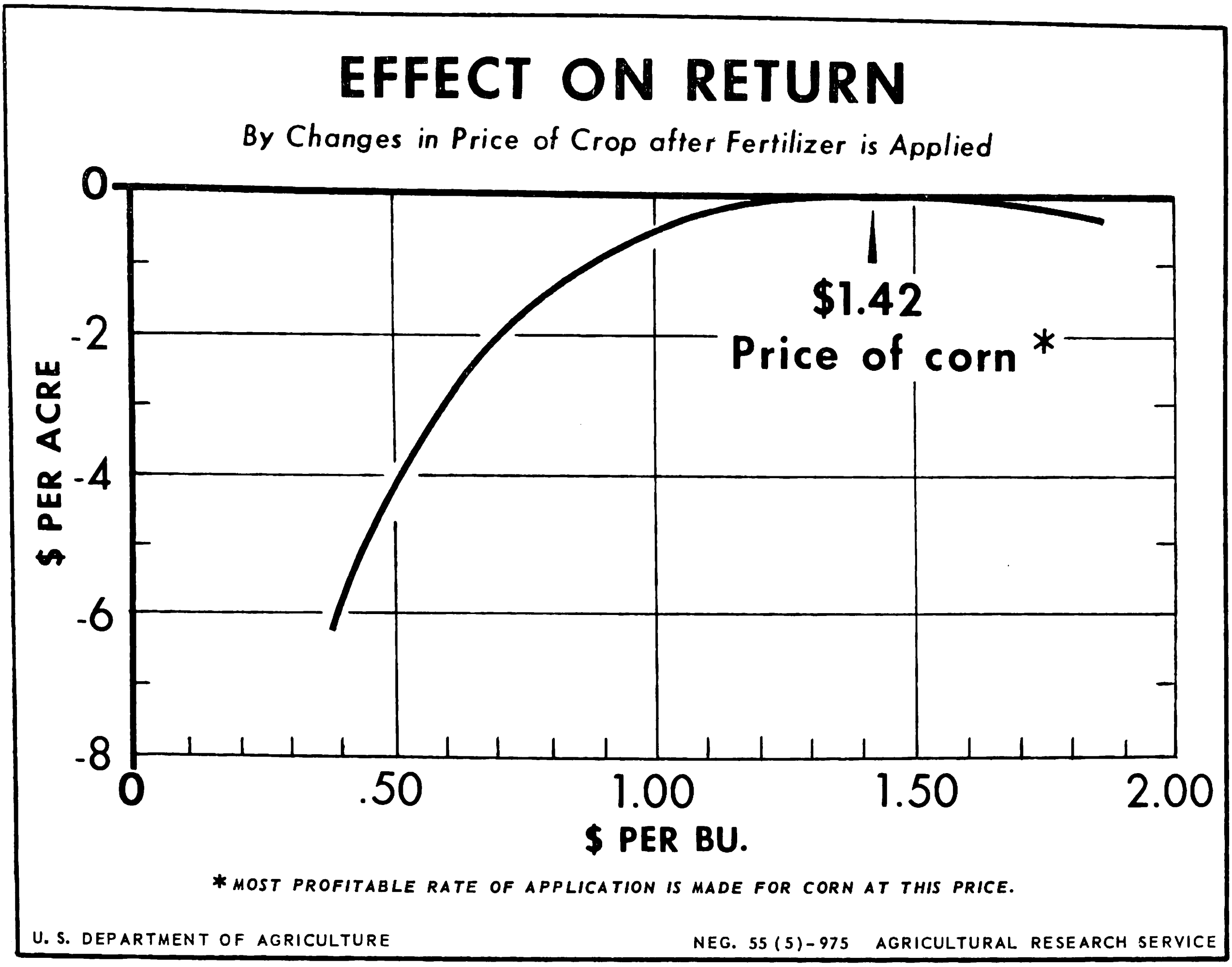

Figure 7.—**Effect on return by changes in price of corn after fertilizer is applied.**

substitutes for 1 unit of P_2O_5. Then, *MRS* in pounds is (0.50288) ($u_a \div u_b$). As u_a is 9.47 pounds and u_b is 7.63 pounds, $u_a \div u_b$ is 1.24115. The marginal rate of substitution in pounds is (0.50288)(1.24115). When allowance is made for rounding figures in the computations, this calculation reflects a substitution of 1 pound of N for 0.625 pound of P_2O_5. This was pointed out earlier as the ratio of the cost per pound of P_2O_5 to the cost per pound of N ($0.10 \div 0.16$).

Using the rate of N shown in column 4, table 3, as the starting point (117 pounds), the associated rate of P_2O_5 required to produce the yield of 118.35 bushels would be 172 pounds. At this combination, the marginal rate of substitution of N for P_2O_5 is about 0.13 pound. Thus, the cost per pound of P_2O_5 would need to be only 13 percent that of N in order to justify this combination for the specified yield. This is not a probable relationship.

Repeating the same calculations, but using 154 pounds of N as the starting point (col. 7, table 3), 109 pounds of P_2O_5 are required to produce the specified yield. Here, the marginal rate of substitution is 2.16 pounds of N per pound of P_2O_5—meaning that the cost per pound of P_2O_5 would need to be 2.16 times that of N in order to justify use of this combination in producing the yield specified.

Based on this experiment, there is only a narrow range within which substitution of 1 nutrient for the other would be economically feasible. Both combinations indicated in the 2 preceding paragraphs are outside that range, as the nutrient-cost ratio associated with either of them is highly improbable. However, many farmers may tend to apply nutrients in relation to their cost per unit without recognizing that the relative rates of response are also important. For example, the average cost of a mixture of 117 pounds of N and 172 pounds of P_2O_5 is about half a cent less per pound than the average cost of 129 pounds of N and 130 pounds of P_2O_5. But the latter is the most profitable combination for the specified price-cost relationship. Both these combinations are calculated to result in the same yield. But at the nutrient-cost ratio used in these calculations, to obtain the same yield the farmer would pay

$2.28 more per acre by using the combination that had the lower weighted average cost per pound. Calculation of equal product combinations may be useful in emphasizing this point.

EXPONENTIAL AND QUADRATIC SQUARE-ROOT EQUATIONS COMPARED

The method described earlier and used in the preceding illustration was applied to two other experiments having the same design and the same rates of application. These involved use of P_2O_5 and K_2O as independent variables, one on alfalfa and one on red clover.[9] Certain results from use of the exponential and a quadratic square-root equation are compared and presented in tables 4 and 5. For the exponential function, the sum of squares explained by regression is based on constants derived for each nutrient in the presence of an application of 160 pounds of the other; except that for the red clover experiment constants for the K_2O curve were developed when P_2O_5 was held at the 80-pound level; and for the alfalfa experiment, constants for the K_2O curve were developed when P_2O_5 was held at the 120-pound level. A better fit was obtained for the K_2O curve in each of these two experiments when P_2O_5 was held constant at the above indicated levels.

TABLE 4.—*Comparison in sums of squared residuals explained by exponential and quadratic square-root equations, as applied to three 9×9 partial factorial experiments*

Experiment	Sums of squared residuals due to treatments	Sums of squared residuals explained by regression		Coefficients of correlation	
		Exponential [1]	Quadratic square root [2]	Exponential [1]	Quadratic square root [2]
Corn	233,811	222,927	222,899	0.9764	0.9764
Alfalfa	26.75	20.78	22.98	.8694	.9229
Red clover	13.66	9.69	11.52	.8425	.9184

[1] Based on constants derived from 17 of the 57 treatment combinations; that is, no convenient simultaneous solution is as yet available for this equation. The sums of squared residuals are computed for all 57 reported treatment mean yields.

[2] Based on constants simultaneously derived, using reported yields for all 57 treatment combinations.

Measures of Fit

The quadratic square-root equation produces[10] a curve that turns down more slowly than the parabola. The parabola is not discussed here.

[9] HEADY, E. O., PESEK, J. T., and BROWN, W. G. CROP RESPONSE SURFACES AND ECONOMIC OPTIMA IN FERTILIZER USE. Iowa Agr. Expt. Sta. Res. Bul. 424, pp. 292–332, illus. 1955.

[10] HEADY, E. O., and PESEK, J. A FERTILIZER PRODUCTION SURFACE WITH SPECIFICATION OF ECONOMIC OPTIMA FOR CORN GROWN ON CALCAREOUS IDA SILT LOAM. Jour. Farm Econ. 36: 466–482, illus. 1954.

The quadratic square-root equation where N and P_2O_5 are the independent variables is written as:

$$y=a+b_1N+b_2P+b_3\sqrt{N}+b_4\sqrt{P}+b_5\sqrt{NP}$$

But, in contrast to the exponential function, which approaches, though theoretically does not reach, the calculated maximum, the quadratic square-root equation provides for calculating reduced yields as more fertilizer is added. If there is merit in being able to calculate reduced yields after the maximum attainable from fertilizer has been reached, the quadratic square-root equation or the parabola would have an advantage over the exponential function. As none of the equations mentioned provides for a phase of increasing increments (sigmoid section), none would appear to have an advantage when fitting to reported yields that may sometimes follow that pattern at low rates of application.

Comparison of the distribution in the deviations of reported yields from those calculated by the two equations is shown in tables 5 and 6. As indicators of reliability of results for use in recommending rates of application, deviations assume importance primarily around the central section of the surface. On soils that are materially deficient in either nutrient, it is impracticable to apply the other unless the level of the deficient nutrient has been raised substantially. If fertilizer is to be used at all, in nearly all cases applications should be made until yields will somewhat exceed those on the steepest part of the curve. This is true because of the substantial returns obtained at applications coincident with the steeper parts of the response curve.

Some farmers have only limited funds. They may, therefore, wish to apply less than calculated most profitable rates. Also, there are possibilities of profitable substitution, in the short run, of applied portions of one nutrient for applied portions of another. However, as pointed out earlier, the range within which it would be profitable to substitute one nutrient for another is limited. This is especially true when the soil content of the nutrient for which a substitution is considered is below the level at which the curve of response to it would begin noticeably to flatten. For this reason, departures from reported yields on the steeper part of the curve do not detract from the validity of recommendations based on a good fit over that part of the curve within which recommendations would fall.

Furthermore, for situations in which the fertility level is such that even the first few yield increments from added fertilizer are quite small, farmers with limited funds ordinarily will use the funds for items other than fertilizer. Therefore, generally speaking, what happens at the "fringe" of the production surface is of relatively little importance. The rate of response far to the left on the curve is often too great to justify limiting applications to that range, even for farmers with limited funds. Far to the right, as the calculated most profitable rate is approached, the response to additional units is so small that risk factors outweigh the possible gains from further incre-

TABLE 5.—*Sums of squared deviations of reported yields in 3 factorial experiments from those calculated, using different equations*

Level of fixed nutrient (pounds)	For regression of—											
	N on P_2O_5		P_2O_5 on K_2O				P_2O_5 on N		K_2O on P_2O_5			
	Corn		Alfalfa		Red clover		Corn		Alfalfa		Red clover	
	Exponential equation	Quadratic square root	Exponential equation	Quadratic square root	Exponential equation	Quadratic square root	Exponential equation	Quadratic square root	Exponential equation	Quadratic square root	Exponential equation	Quadratic square root
0	549	1,141	1.9249	0.5138	0.5252	0.1711	950	970	0.9802	0.5559	0.1310	0.1013
40	514	239	.3121	.2313	.1703	.1263	1,230	948	.2578	.2307	.1690	.1174
80	531	522	.0271	.1096	.1130	.0594	245	208	.1568	.0885	.0762	.0971
120	670	254	.2455	.3011	.1358	.1370	464	687	.0505	.1127	.3014	.3885
160	409	619	.0739	.0958	.0884	.0915	738	641	.3277	.1385	.4558	.1965
200	850	806	.1189	.1710	.1171	.0512	560	618	.3983	.2096	.1042	.0490
240	460	317	.1680	.0958	.1372	.0669	153	356	.4553	.1192	.2785	.1073
280	317	592	.1240	.0877	.2528	.3852	215	236	.2327	.2244	.0523	.0496
320	1,142	995	.2710	.1882	.4418	.3141	887	821	.4061	.0948	.4132	.2958
Total	5,442	5,485	3.2654	1.7743	1.9816	1.4025	5,442	5,485	3.2654	1.7743	1.9816	1.4025

TABLE 6.—*Sums of squared deviations of reported yields in 3 factorial experiments from yields calculated at all levels of each of the two nutrients except 0 and 320 pounds, using different equations*

Level of fixed nutrient (pounds)	For regression of—											
	N on P_2O_5		P_2O_5 on K_2O				P_2O_5 on N		K_2O on P_2O_5			
	Corn		Alfalfa		Red clover		Corn		Alfalfa		Red clover	
	Exponential equation	Quadratic square root	Exponential equation	Quadratic square root	Exponential equation	Quadratic square root	Exponential equation	Quadratic square root	Exponential equation	Quadratic square root	Exponential equation	Quadratic square root
40	236	217	0.1618	0.2018	0.0731	0.0788	469	545	0.1592	0.1789	0.0059	0.0300
80	412	404	.0253	.0650	.1119	.0544	113	139	.1201	.0796	.0559	.0295
120	218	65	.2180	.2507	.1137	.1202	234	393	.0333	.0495	.2643	.3197
160	387	450	.0731	.0522	.0619	.0378	733	458	.0919	.1075	.2083	.1110
200	682	779	.1085	.1039	.0325	.0162	478	427	.1262	.1433	.0031	.0431
240	125	185	.1236	.0887	.0271	.0198	82	317	.0408	.0548	.0767	.0526
280	236	266	.0357	.0463	.2224	.2968	187	87	.1745	.1950	.0284	.0381
Total	2,296	2,366	.7460	.8086	.6426	.6240	2,296	2,366	.7460	.8086	.6426	.6240

ments in yield, even though a farmer may have unlimited funds. Thus, the principal function of a yield equation is to estimate most profitable rates and combinations of the different independent variables. To do this efficiently requires an adequate number of rates in the portion of greatest curvature and in the area of the maximum yield. Other practical factors, such as knowledge of probabilities of different types of risk and estimates of relative returns from different inputs that may compete for the farmer's dollar, are taken into account in deciding how much to apply.

Some further statistics that bear on this discussion are presented in table 6. This table shows deviations from regression if deviations at the 0- and the 320-pound rates of both nutrients in each case are omitted. Good experimental procedure requires the use of rates somewhat higher than are expected to result in further increases in yield if one is to be certain that practical maximum response has been attained for the conditions. It is also necessary that rates be carried to such levels to insure reliability of constants used in the prediction equations, whether graphic or more precise methods are used.

Results, as applied to the three experiments noted here, are particularly useful in bringing out the point that considerable segments of the curve (the steep and the flat portions) are outside the range of rate of application that would be recommended. The soil content of two of the nutrients included in each case—N and P_2O_5—was extremely low, so that yields at low applications fell on steep sections of the response curve. Also, the higher rates applied resulted in little or no further increase in yields. Possibly other limiting factors were present, in the absence of which response would have been obtained at even higher rates than were used.

Referring back to table 4, the coefficients of correlation were the same for both equations in the corn experiment, but they were higher for the quadratic square-root equation as applied to the alfalfa and red clover experiments. These coefficients are based on deviation explained by regression over the complete surface. But table

6 shows that if attention is centered only on the parts of the surface that include rates which would ordinarily be recommended, coefficients of correlation would be slightly higher for the exponential equation for the corn and the alfalfa experiments. For the red clover experiment the quadratic square-root equation would have a slight advantage.

Even when the quadratic square-root equation is used, more than half (54 to 57 percent) of all deviations from regression in the 3 experiments are accounted for by deviations at the lowest and the highest rates of both nutrients applied in the experiment. These deviations are those around the fringe of the production surface. Using the exponential function, 58 percent of the deviation from regression occurred in this fringe area for the corn experiment, 68 percent for the red clover experiment, and 77 percent for the alfalfa experiment. The quadratic square-root equation appears to fit the fringe areas better than the exponential function as applied to the alfalfa and the red clover experiments. But the rest of the surface for the corn and alfalfa experiments was a little better fitted by the exponential function. Deviations from regression in the red clover experiment were slightly smaller for the rest of the surface when the quadratic square-root equation was used. Table 6 shows that when deviations from regression for the portions of the surface outside the range of recommendations are deducted, there is no appreciable difference in the fit obtained through use of the 2 equations as applied to these 3 experiments.

Most Profitable Rates and Confidence Limits

Differences in most profitable rates as calculated, using the 2 equations, are shown in table 7 for the 3 experiments. The differences are not important for the purpose of using the results as a basis for recommendations. Differences in yields are small, even though there are substantial differences in most profitable rates resulting from use of the 2 equations. For the alfalfa and clover experiments the differences in returns per acre above cost of fertilizer are negligible. For the corn experiment, the difference is about $6.00 per acre, which may be regarded as small when viewed in the light of probabilities of variation in yield because of weather and other risks. Also, this difference is only half the value of 1 standard error in yield at the most profitable rate, as calculated from the quadratic square-root equation.

The unimportance of the differences in results obtained with the 2 equations is further brought out by comparing yields and returns per acre in table 7 with those of table 8. For purposes of table 8, the most profitable rates derived from using each equation are substituted in the formula of the other. Comparison of the 2 tables shows that the differences in results are too small to affect any decisions the farmer might make. The greatest difference in return above cost of fertilizer occurs in the corn experiment—$128.30 in table 7 versus $140.59 in table 8. However, this difference is approximately the same as the value of 1 standard error in yield at the most profitable rate.

Confidence limits in yield at the most profitable combination have not been calculated for the exponential equation because of lack of a convenient method for simultaneous solution of values of the constants. Comparison of the residual sums of squares for the whole surface and for the more relevant part of the surface was presented earlier. For the quadratic square-root equation at the 67-percent level of probability, confidence limits in yields at the most profitable rates in the three experiments are ± 8.60 bushels of corn, ± 0.184 ton of alfalfa, and ± 0.077 ton of red clover. In terms of dollar returns per acre above cost of fertilizer, these limits are $\pm \$12.21$ for corn, $2.76 for alfalfa, and $1.16 for red clover. Only in the case of the corn

TABLE 7.—*Comparison of most profitable combinations of plant nutrients, yields at these combinations, and returns above cost of fertilizer, as calculated by 2 equations applied to three 9×9 partial factorial experiments*

Item	Experiment					
	Corn		Alfalfa		Red clover	
	Exponential	Quadratic square root	Exponential	Quadratic square root	Exponential	Quadratic square root
Most profitable rate of—						
N [1] ... pounds	129	160	---	---	---	---
P_2O_5 ... do	130	173	50	51	40	95
K_2O [2] ... do	---	---	56	23	11	5
Total nutrients ... do	259	333	106	74	51	100
Yield ... bu. or tons	118.35	120.56	3.18	3.08	3.02	3.38
Price per unit of crop ... dollars	1.42	1.42	15.00	15.00	15.00	15.00
Value of crop per acre ... do	168.06	171.20	47.66	46.20	45.30	50.70
Cost of fertilizer ... do	33.64	42.90	8.36	6.48	4.66	9.80
Returns per acre above cost of fertilizer ... do	134.42	128.30	39.40	39.78	40.64	40.90

[1] N not a variable in alfalfa and red clover experiments.
[2] K_2O not a variable in corn experiment.

experiment is the confidence limit possibly greater than the margin of error in attempting to apply the most profitable rate under farm conditions.

Were methods available to calculate comparable confidence limits associated with use of the two-variable form of the exponential equation, they might be expected to be of about the same order of magnitude as those indicated above. This expectation is based on comparison of residual sums of squares for the entire production surface and for that remaining after eliminating from consideration a part of it that is of no practical importance.

TABLE 8.—*Yields and returns per acre, if calculated by substituting most profitable rates derived from each equation into the formula of the other*

Item	Substituting most profitable rates derived—					
	From quadratic square-root equation into exponential formula			From exponential equation into quadratic square-root formula		
	Corn	Alfalfa	Red clover	Corn	Alfalfa	Red clover
Yield.....bu. or tons..	122.57	2.99	3.20	122.70	3.16	3.18
Value.....dollars..	174.05	44.85	48.00	174.23	47.40	47.70
Cost of fertilizer.....do....	42.90	6.48	9.80	33.64	8.36	4.66
Return per acre above cost of fertilizer.....do....	131.15	38.37	38.20	140.59	39.04	43.04

APPENDIX

Calculation of Most Profitable Rate of One Nutrient

On page 9, reference was made to calculation of the most profitable rate of one nutrient rather than estimation directly from the curve, as shown in figure 4. The graphic method requires that small units of application, such as 10 pounds, be indicated on the base scale of the overlay sheet so that increments in yield can be estimated visually from the curve. Small units of application and corresponding small increments in yield are necessary if the graphic estimate of the most profitable rate is to be close to the accurately calculated rate. Those familiar with use of a machine may find it easier to calculate the most profitable rate after fitting the curve by graphic methods.

A simple way of doing this involves use of a factor Q, which is calculated as $4.48142 \div vM$ when 4.48142 is a constant for this computation,[11] v is the price per unit of the crop before harvest, and M is the theoretical maximum yield. Referring to figure 3, M is 126.4 bushels and on figure 4, v is \$1.40. Thus: $Q = 4.48142 \div 1.40(126.4) = 0.02532$ then

$Qr' = 0.02532(1.52) = 0.03849 = R^{n+a}$ and

$1 - R^{n+a} = 0.96151$ at the most profitable rate, thus,

$n + a = 14.60$ units (from table of $1 - R^x$ values), and as

$n = 0.90$ unit (from fig. 3 or by optional method described)

$a = 13.70$ units or

$a = 129.7$ pounds (a in units times u_a, or 9.47)

$y = M(1 - R^{n+a}) = 126.4(0.96151) = 121.5$ bushels

This rate, 129.7 pounds, and this yield, 121.5 bushels, are substantially the same as the graphic estimates shown in figure 3.

Calculation of Economic Combinations of Three Nutrients

The most economic combinations of N, P_2O_5, and K_2O at specified costs per pound of the nutrients, prices of the crop required to render each combination most profitable, and yields and returns at each combination, are calculated as illustrated in table 3 (p. 10), in which only two nutrients were involved. The method of handling the clerical steps after graphically fitting the curves is outlined in form 1 (see p. 19). This form is self-explanatory, as all terms used have been defined and illustrated.

Calculation of the Most Profitable Combination of Nutrients

Table 3 of the text and form 1 illustrate calculation of a series of combinations of 2 or of 3 nutrients, respectively, that would be most economical when their costs have been specified. With this method, the price of the crop at which each of these combinations would be most profitable is derived.

If only the combination is used that would be most profitable when both cost of fertilizer and price of the crop are specified, steps in the calculation are different. Form 2 (p. 19) shows these steps when 2 nutrients are involved; form 3 (p. 19) when all 3 nutrients are variables.

Form 3 represents a short-cut method of calculating the most profitable combination of 3 nutrients that produces results generally accurate enough for practical purposes. Exact determination of the most profitable combination involves an iterative technique that is avoided in the method presented here.

When 3 nutrients are involved, experience indicates that if the value of $p+b$, item 25, form 3, turns out to be lower than 10 to 14 units, the answers obtained may contain enough error to make a practical difference. However, a test of this is readily made by calculating a factor T as

$$\frac{\text{(Item 23) (Item 24) (Item 28)}}{\text{(Item 31) (Item 32)}}$$

If T is found to differ from Q by no more than about 0.0005, the calculated most profitable combination will be substantially correct.

If a second trial is necessary, a value may be substituted for $p+b$ that is about the same percentage above or below the first value found as the percentage by which T exceeds or is smaller than Q. When a new value is substituted for $p+b$, item 25, form 3, a new associated value for item 24 is read from a table of $1-R^x$ values. Then 1.0 minus this new $1-R^{p+b}$ value, item 24, will result in a new value for item 23, which is R^{p+b}. No adjustments need to be made back of this point. The test may then be repeated if necessary to render $T-Q$ no greater than about ± 0.0005.

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[11] $4.48142 = \frac{1}{0.22314} = \frac{1}{(-ln0.8)}$

FORM 1.—*Steps in calculating economic combinations of N, P_2O_5, and K_2O at specified costs per pound of the nutrients, prices of the crop required to render each combination most profitable, and yields and returns at each combination*

Item and description or derivation	(1)	(2)	(3)	(4)
1. b, or applied P_2O_5 ---------- pounds				
2. b in units [item 1 divided by u_b; u_b equals ___ pounds] ---------- units				
3. $p+b$ [item 2 plus (p); p equals ___] ---- do				
4. $1-R^{p+b}$ [read from standard yield curve or from table 9] ---------- decimals				
5. R^{p+b} [1.0 minus item 4] ---------- do				
6. r' (R^{p+b}) [r' equals cost per pound N (__cents) times u_a (__pounds), or $__] ---- numbers				
7. r'' ($1-R^{p+b}$) [r'' equals cost per pound P_2O_5 (__cents) times u_b (__pounds), or $__] ---------- numbers				
8. Item 6 plus item 7 ---------- do				
9. R^{n+a} [item 6 divided by item 8] ---- decimals				
10. $1-R^{n+a}$ [1.0 minus item 9] ---------- do				
11. $n+a$ [read from standard yield curve or from table 9] ---------- units				
12. a [item 11 minus n; n equals __] ---------- do				
13. Most profitable rate of N to apply [item 12 times u_a; u_a equals __pounds] ---- pounds				
14. r''' (R^{p+b}) [r''' times item 5; r''' equals $__] numbers				
15. Item 7 plus item 14 ---------- do				
16. R^{k+c} [item 14 divided by item 15] -- decimals				
17. $1-R^{k+c}$ [1.0 minus item 16] ---------- do				
18. $k+c$ [read from standard yield curve or from table 9] ---------- units				
19. c [item 18 minus k; k equals __] ---------- do				
20. Most profitable rate of K_2O to apply [item 19 times u_c; u_c equals __pounds] ---- pounds				
21. (Item 4) (item 5) (item 7) divided by (item 8) (item 15) ---------- decimals				
22. v [1.0 divided by item 21 (0.22314) (M_3); M_3 equals __] ---------- dollars				
23. y [M_3 (item 4) (item 10) (item 17)] ---- yield				
24. Gross returns [item 23 times price] [1] -- dollars				
25. Cost of fertilizer [r' (item 12) plus r'' (item 2) plus r''' (item 19)] ---------- dollars				

[1] It may be desirable to calculate the gross return for each combination at the current price rather than at v, which is the price required to render a combination the most profitable one. v, item 22, is different for each combination. Remaining steps as desired may be calculated in same way as items 3 to 27, table 3.

FORM 2.—*Most profitable combination of 2 nutrients (N and P_2O_5)*

Crop__________ Unit of yield__________

Source of data__________

$v=\$$______ $M_2=$______ $Q=4.48142\div vM_2$, or______

r'=cost per pound of N times u_a, or $______

r''=cost per pound of P_2O_5 times u_b, or $______

Item	Value
1. Qr''	
2. $-Qr'$	
3. Item 1 plus item 2	
4. −1.0 plus item 3 with changed sign	
5. One-half of item 4 [1]	
6. Item 5 squared [2]	
7. Item 6 minus item 1	
8. Square root of item 7	
9. R^{p+b} [item 5 minus item 8]	
10. $1-R^{p+b}$ [1.0 minus item 9]	
11. $p+b$ [from standard curve or from table 9]	
12. b in units [item 11 minus p; p equals______]	
13. b in pounds [item 12 times u_b; u_b equals______]	
14. R^{n+a} [item 2 divided by item 10] [1]	
15. $1-R^{n+a}$ [1.0 minus item 14]	
16. $n+a$ [from standard curve or from table 9]	
17. a in units [item 16 minus n, n equals______]	
18. a in pounds [item 17 times u_a; u_a equals______]	

[1] Drop minus sign.

[2] If item 6 is smaller than item 1, it means that a solution is not possible by this shortcut method and the method of table 3 is suggested.

FORM 3.—*Most profitable combination of 3 nutrients (N, P_2O_5, and K_2O)*

Area__________ Crop__________ Unit of yield__________ Source of data__________

$v=\$$__________ $M_3=$__________ $Q=4.48142\div vM_3$, or__________

Item	Value	Item	Value
1. $-2.0\,r''$ [r'' equals $____]		22. Square root of item 21	
2. $r'-r''$ [r' equals $____]		23. Item 19 minus item 22	
3. $r'''-r''$ [r''' equals $____]		24. 1.0 minus item 23	
4. r'' squared		25. $p+b$ [from table 9]	
5. Item 2 times r''		26. b [item 25 minus p; p equals ____]	
6. Item 3 times r''		27. Item 26 times u_b [u_b equals ____ pounds]	
7. Item 5 plus item 6		28. Item 24 times r''	
8. Item 2 times item 3		29. Item 23 times r'	
9. Item 4 times Q		30. Item 23 times r'''	
10. Item 7 times Q		31. Item 28 plus item 29	
11. Item 8 times Q		32. Item 28 plus item 30	
12. r'' minus item 10		33. Item 29 divided by item 31	
13. Item 1 minus item 11		34. 1.0 minus item 33	
14. Item 13 divided bv r''		35. $n+a$ [from table 9]	
15. Item 12 divided by r''		36. a [item 35 minus n; n equals ____]	
16. Item 9 divided by r''		37. Item 36 times u_a [u_a equals ____ pounds]	
17. Item 15 divided by item 14		38. Item 30 divided by item 32	
18. Item 16 divided by item 14		39. 1.0 minus item 38	
19. Item 17 divided by 2.0 [drop minus sign]		40. $k+c$ [from table 9]	
20. Item 19 squared		41. c [item 40 minus k; k equals ____]	
21. Item 18 plus item 20 [1]		42. Item 41 times u_c [u_c equals ____ pounds]	

[1] If item 21 is negative, it means that a solution is not possible by this method and the method of form 1 is suggested.

Values of $1-R^x$

TABLE 9.—*Values of* $1-R^x$ *when* $R=0.8$

(All values are decimal fractions)

x	0	1	2	3	4	5	6	7	8	9
0.00	----------	000223	000446	000669	000892	001125	001338	001561	001784	002006
01	002229	002452	002674	002897	003119	003342	003564	003786	004009	004231
02	004453	004675	004897	005119	005341	005563	005785	006007	006228	006450
03	006672	006894	007115	007337	007558	007780	008001	008222	008444	008665
04	008886	009107	009328	009549	009770	009991	010212	010433	010654	010874
05	011095	011315	011536	011757	011977	012198	012418	012639	012859	013079
06	013299	013519	013740	013960	014180	014400	014620	014839	015059	015279
07	015499	015718	015938	016158	016377	016597	016816	017035	017254	017474
08	017693	017912	018131	018350	018569	018788	019007	019226	019445	019664
09	019883	020101	020320	020538	020757	020975	021194	021412	021631	021849
10	022067	022285	022503	022721	022940	023158	023376	023593	023811	024029
11	024247	024465	024683	024900	025118	025335	025553	025770	025987	026205
12	026422	026639	026856	027073	027290	027508	027725	027942	028158	028375
13	028592	028809	029026	029242	029459	029675	029892	030108	030325	030541
14	030757	030974	031190	031406	031622	031838	032054	032270	032486	032702
15	032918	033133	033349	033565	033780	033996	034212	034427	034642	034858
16	035073	035288	035504	035719	035934	036149	036364	036579	036794	037009
17	037224	037439	037654	037868	038083	038298	038512	038726	038941	039156
18	039370	039584	039799	040013	040227	040441	040655	040869	041083	041297
19	041511	041725	041939	042153	042366	042580	042794	043007	043220	043434
20	043648	043861	044074	044288	044501	044714	044927	045140	045353	045566
21	045779	045992	046205	046418	046631	046843	047055	047268	047481	047794
22	047906	048119	048331	048543	048755	048968	049180	049392	049604	049816
23	050028	050240	050452	050664	050876	051088	051298	051511	051723	051934
24	052145	052357	052569	052780	052991	053203	053414	053625	053836	054047
25	054258	054469	054680	054891	055102	055313	055524	055735	055945	056156
26	056366	056577	056787	056998	057208	057419	057629	057839	058049	058260
27	058470	058680	058890	059100	059310	059520	059729	059939	060149	060358
28	060568	060778	060987	061197	061406	061616	061825	062035	062244	062453
29	062662	062872	063081	063290	063499	063707	063916	064125	064334	064543
30	064752	064960	065169	065377	065586	065794	066003	066211	066420	066628
31	066836	067044	067253	067461	067669	067877	068085	068293	068500	068708
32	068916	069124	069332	069540	069747	069954	070162	070369	070577	070784
33	070992	071199	071406	071613	071820	072027	072234	072441	072648	072855
34	073062	073269	073476	073683	073889	074096	074302	074509	074715	074921
35	075128	075335	075541	075747	075953	076160	076366	076572	076778	076984
36	077190	077396	077602	077807	078013	078219	078424	078630	078836	079041
37	079247	079452	079658	079863	080068	080273	080478	080684	080889	081094
38	081299	081504	081709	081914	082119	082323	082528	082732	082937	083142
39	083347	083551	083756	083960	084165	084369	084573	084777	084982	085186
40	085390	085594	085798	086002	086206	086410	086613	086817	087021	087225
41	087429	087632	087836	088039	088243	088446	088649	088853	089056	089259
42	089463	089666	089869	090072	090275	090478	090681	090884	091086	091289
43	091492	091695	091898	092100	092303	092505	092708	092910	093113	093315
44	093517	093719	093922	094124	094326	094528	094730	094932	095134	095336
45	095538	095740	095941	096143	096345	096546	096748	096949	097151	097352
46	097554	097755	097956	098157	098359	098560	098761	098962	099163	099364
47	099565	099766	099967	100168	100369	100569	100770	100970	101171	101371
48	101572	101773	101973	102173	102374	102574	102774	102974	103175	103375
49	103575	103775	103975	104175	104374	104574	104774	104974	105173	105373
50	105573	105772	105972	106171	106371	106570	106770	106969	107168	107367
51	107567	107766	107965	108164	108363	108562	108760	108959	109158	109357
52	109556	109754	109953	110152	110350	110549	110747	110945	111143	111342
53	111540	111738	111937	112135	112333	112531	112729	112927	113125	113323
54	113521	113719	113916	114114	114312	114509	114707	114904	115102	115299
55	115497	115694	115891	116089	116286	116483	116680	116877	117074	117271
56	117468	117665	117862	118059	118256	118452	118649	118846	119042	119239
57	119435	119632	119828	120024	120220	120417	120614	120810	121006	121202
58	121398	121594	121790	121986	122182	122378	122574	122769	122965	123160
59	123356	123552	123748	123943	124138	124334	124529	124725	124920	125115
60	125310	125505	125701	125896	126091	126286	126481	126676	126870	127065
61	127260	127455	127649	127844	128039	128233	128428	128622	128817	129011
62	129206	129400	129594	129788	129982	130176	130370	130564	130758	130952
63	131146	131340	131534	131728	131921	132115	132308	132502	132696	132889
64	133083	133276	133470	133663	133856	134050	134243	134436	134629	134822
65	135015	135208	135401	135594	135787	135980	136172	136365	136558	136750
66	136943	137136	137328	137521	137713	137906	138098	138290	138483	138675
67	138867	139059	139251	139443	139635	139827	140019	140211	140403	140595
68	140786	140978	141170	141361	141553	141744	141936	142127	142319	142510
69	142701	142893	143084	143275	143466	143657	143848	144039	144230	144421
70	144612	144803	144994	145185	145376	145566	145757	145947	146138	146328
71	146519	146709	146900	147090	147280	147471	147661	147851	148041	148231
72	148421	148611	148801	148991	149181	149371	149561	149750	149940	150130
73	150319	150509	150698	150888	151077	151267	151456	151645	151835	152024
74	152213	152403	152592	152781	152970	153159	153348	153536	153725	153914
75	154103	154292	154480	154669	154858	155046	155235	155423	155612	155800
76	155988	156177	156365	156553	156741	156930	157118	157306	157494	157682
77	157870	158058	158245	158433	158621	158809	158997	159184	159372	159559
78	159747	159934	160122	160309	160497	160684	160871	161058	161245	161433
79	161620	161807	161994	162181	162368	162555	162741	162928	163115	163302
80	163488	163675	163862	164048	164235	164421	164607	164794	164980	165167
81	165353	165539	165725	165912	166098	166284	166470	166656	166841	167027
82	167213	167399	167585	167771	167956	168142	168327	168513	168698	168884
83	169070	169255	169440	169625	169811	169996	170181	170366	170552	170737
84	170922	171107	171292	171476	171661	171846	172031	172216	172400	172585
85	172770	172954	173139	173323	173508	173692	173876	174061	174245	174429
86	174614	174798	174982	175166	175350	175534	175718	175902	176085	176269
87	176453	176637	176821	177004	177188	177371	177555	177738	177922	178105
88	178289	178472	178655	178839	179022	179205	179388	179571	179754	179937
89	180120	180303	180486	180669	180852	181035	181217	181400	181583	181765

TABLE 9.—*Values of $1-R^x$ when $R=0.8$*—Continued

(All values are decimal fractions)

x	0	1	2	3	4	5	6	7	8	9
0.90	181948	182130	182313	182495	182678	182860	183042	183225	183407	183589
91	183771	183953	184135	184317	184499	184681	184863	185045	185227	185409
92	185591	185772	185954	186136	186317	186499	186680	186862	187043	187225
93	187406	187587	187768	187950	188131	188312	188493	188674	188855	189036
94	189217	189398	189579	189760	189940	190121	190302	190483	190663	190844
95	191024	191205	191385	191566	191746	191926	192107	192287	192467	192647
96	192827	193008	193188	193368	193548	193728	193907	194087	194267	194447
97	194627	194806	194986	195166	195345	195525	195704	195884	196063	196242
98	196422	196601	196780	196960	197139	197318	197497	197676	197855	198034
99	198213	198392	198571	198750	198928	199107	199286	199464	199643	199821
1.00	200000	200179	200357	200535	200714	200892	201070	201249	201427	201605
01	201783	201961	202139	202317	202495	202673	202851	203029	203207	203385
02	203563	203740	203918	204095	204273	204451	204628	204805	204983	205160
03	205338	205515	205692	205869	206047	206224	206401	206578	206755	206932
04	207109	207286	207463	207640	207816	207993	208170	208346	208523	208700
05	208876	209053	209229	209406	209582	209758	209935	210111	210287	210463
06	210640	210816	210992	211168	211344	211520	211696	211872	212047	212223
07	212399	212575	212750	212926	213102	213277	213453	213628	213804	213979
08	214154	214330	214505	214680	214856	215031	215206	215381	215556	215731
09	215906	216081	216256	216431	216606	216780	216955	217130	217304	217479
10	217654	217828	218003	218177	218352	218526	218700	218875	219049	219223
11	219398	219572	219746	219920	220094	220268	220442	220616	220790	220964
12	221138	221311	221485	221659	221832	222006	222180	222353	222526	222700
13	222874	223047	223220	223394	223567	223740	223913	224086	224260	224433
14	224606	224779	224952	225125	225298	225470	225643	225816	225989	226161
15	226334	226507	226679	226852	227024	227197	227369	227542	227714	227886
16	228059	228231	228403	228575	228747	228919	229091	229263	229435	229607
17	229779	229951	230123	230295	230466	230638	230810	230981	231153	231324
18	231496	231667	231839	232010	232182	232353	232524	232695	232867	233038
19	233209	233380	233551	233722	233893	234064	234235	234406	234577	234747
20	234918	235089	235260	235430	235601	235771	235942	236112	236282	236453
21	236623	236794	236964	237134	237304	237475	237645	237815	237985	238155
22	238325	238495	238665	238835	239004	239174	239344	239513	239683	239853
23	240023	240192	240362	240531	240701	240870	241039	241209	241378	241547
24	241717	241886	242055	242224	242393	242562	242731	242900	243069	243238
25	243407	243576	243744	243913	244082	244250	244419	244588	244756	244925
26	245093	245262	245430	245598	245767	245935	246103	246271	246440	246608
27	246776	246944	247112	247280	247448	247616	247784	247951	248119	248287
28	248455	248622	248790	248958	249125	249293	249460	249628	249795	249963
29	250130	250297	250464	250632	250799	250966	251133	251300	251467	251634
30	251801	251968	252135	252302	252469	252635	252802	252969	253135	253302
31	253469	253635	253802	253968	254135	254301	254468	254634	254800	254967
32	255133	255299	255465	255631	255797	255963	256130	256296	256461	256627
33	256793	256959	257125	257291	257456	257622	257788	257953	258119	258284
34	258450	258615	258781	258946	259111	259277	259442	259607	259772	259938
35	260103	260268	260433	260598	260763	260928	261093	261257	261422	261587
36	261752	261917	262081	262246	262410	262575	262740	262904	263068	263233
37	263397	263562	263726	263890	264055	264219	264383	264547	264711	264875
38	265039	265203	265367	265531	265695	265859	266022	266186	266350	266514
39	266677	266841	267005	267168	267332	267495	267659	267822	267985	268149
40	268312	268475	268638	268801	268964	269128	269291	269454	269617	269780
41	269943	270106	270269	270431	270594	270757	270920	271082	271245	271407
42	271570	271733	271895	272057	272220	272382	272545	272707	272869	273031
43	273194	273356	273518	273680	273842	274004	274166	274328	274490	274652
44	274814	274976	275137	275299	275461	275622	275784	275945	276107	276269
45	276430	276592	276753	276914	277076	277237	277398	277559	277721	277882
46	278043	278204	278365	278526	278687	278848	279009	279170	279331	279491
47	279652	279813	279973	280134	280295	280455	280616	280776	280937	281097
48	281258	281418	281578	281739	281899	282059	282219	282380	282540	282700
49	282860	283020	283180	283340	283500	283660	283819	283979	284139	284299
50	284458	284618	284777	284937	285097	285256	285415	285575	285734	285894
51	286053	286212	286372	286531	286690	286849	287008	287167	287326	287486
52	287644	287803	287962	288121	288280	288439	288598	288756	288915	289074
53	289232	289391	289550	289708	289867	290025	290183	290342	290500	290658
54	290816	290975	291133	291291	291449	291607	291765	291923	292081	292239
55	292397	292555	292713	292871	293029	293186	293344	293502	293659	293817
56	293975	294132	294290	294447	294604	294762	294919	295077	295234	295391
57	295548	295706	295863	296020	296177	296334	296491	296648	296805	296962
58	297118	297275	297432	297589	297746	297902	298059	298216	298372	298528
59	298685	298841	298998	299154	299311	299467	299623	299780	299936	300093
60	300249	300405	300561	300717	300873	301029	301185	301341	301496	301652
61	301808	301964	302119	302275	302431	302587	302742	302898	303053	303209
62	303364	303520	303675	303830	303986	304141	304296	304452	304607	304762
63	304917	305072	305227	305382	305537	305692	305847	306002	306157	306312
64	306466	306621	306776	306931	307085	307240	307394	307549	307703	307858
65	308012	308166	308321	308475	308629	308784	308938	309092	309246	309401
66	309555	309709	309863	310017	310171	310325	310478	310632	310786	310940
67	311093	311247	311401	311555	311708	311862	312015	312169	312322	312476
68	312629	312782	312936	313089	313242	313396	313549	313702	313855	314008
69	314161	314314	314467	314620	314773	314926	315079	315232	315385	315537
70	315690	315843	315995	316148	316300	316453	316605	316758	316910	317063
71	317215	317367	317520	317672	317824	317977	318129	318281	318433	318585
72	318737	318889	319041	319193	319345	319497	319648	319800	319952	320104
73	320256	320407	320559	320711	320862	321014	321165	321317	321468	321619
74	321771	321923	322074	322225	322376	322527	322678	322829	322980	323131
75	323282	323433	323584	323735	323886	324037	324188	324339	324489	324640
76	324791	324941	325092	325243	325393	325544	325694	325845	325995	326145
77	326296	326446	326596	326746	326897	327047	327197	327347	327497	327648
78	327798	327947	328097	328247	328397	328547	328697	328847	328997	329146
79	329296	329445	329595	329744	329894	330044	330193	330343	330492	330641

Table 9.—*Values of* $1-R^x$ *when* $R=0.8$—Continued

(All values are decimal fractions)

x	0	1	2	3	4	5	6	7	8	9
1.80	330791	330940	331089	331238	331388	331537	331686	331835	331984	332133
81	332282	332431	332580	332729	332878	333027	333176	333325	333474	333622
82	333771	333919	334068	334217	334365	334514	334762	334810	334959	335107
83	335256	335404	335552	335701	335849	335997	336145	336293	336441	336589
84	336737	336885	337033	337181	337329	337477	337625	337773	337920	338068
85	338216	338363	338511	338659	338806	338954	339101	339249	339396	339543
86	339691	339838	339985	340133	340280	340427	340574	340721	340868	341015
87	341163	341310	341457	341604	341750	341897	342044	342191	342338	342484
88	342631	342778	342924	343071	343218	343364	343511	343657	343803	343950
89	344096	344243	344389	344535	344681	344828	344974	345120	345266	345412
90	345558	345704	345850	345996	346142	346288	346434	346580	346726	346872
91	347017	347163	347308	347454	347600	347745	347891	348036	348182	348327
92	348472	348618	348763	348909	349054	349199	349344	349489	349634	349780
93	349925	350070	350215	350360	350505	350650	350794	350939	351084	351229
94	351374	351518	351663	351808	351952	352097	352241	352385	352530	352675
95	352820	352964	353108	353252	353397	353541	353685	353829	353974	354118
96	354262	354406	354550	354694	354838	354982	355126	355270	355414	355558
97	355701	355845	355989	356132	356276	356420	356563	356707	356851	356994
98	357137	357281	357424	357568	357711	357854	357998	358141	358284	358427
99	358570	358713	358856	359000	359143	359286	359428	359571	359714	359857
2.00	360000	360143	360285	360428	360571	360714	360856	360999	361141	361284
01	361426	361569	361711	361854	361996	362139	362281	362423	362565	362708
02	362850	362992	363134	363276	363418	363560	363702	363844	363986	364128
03	364270	364412	364554	364695	364837	364979	365121	365262	365404	365545
04	365687	365829	365970	366112	366253	366394	366536	366677	366818	366960
05	367101	367242	367383	367525	367666	367807	367948	368089	368230	368371
06	368512	368652	368793	368934	369075	369216	369357	369497	369638	369778
07	369919	370060	370200	370341	370481	370622	370762	370903	371043	371183
08	371323	371464	371604	371744	371884	372025	372165	372305	372445	372585
09	372725	372865	373005	373145	373285	373424	373564	373704	373844	373983
10	374123	374263	374402	374542	374681	374821	374960	375100	375240	375379
11	375518	375657	375797	375936	376075	376214	376354	376493	376632	376771
12	376910	377049	377188	377327	377466	377605	377744	377882	378021	378160
13	378299	378438	378576	378715	378854	378992	379131	379269	379408	379546
14	379685	379823	379961	380100	380238	380376	380515	380653	380791	380929
15	381067	381205	381343	381481	381619	381757	381895	382033	382171	382309
16	382447	382585	382722	382860	382998	383135	383273	383411	383548	383686
17	383823	383961	384098	384236	384373	384510	384648	384785	384922	385060
18	385197	385334	385471	385608	385745	385882	386019	386156	386293	386430
19	386567	386704	386841	386978	387114	387251	387388	387524	387661	387798
20	387934	388071	388207	388344	388480	388617	388753	388889	389026	389162
21	389299	389435	389571	389707	389843	389980	390116	390252	390388	390524
22	390660	390796	390932	391068	391204	391339	391475	391611	391747	391883
23	392018	392154	392289	392425	392561	392696	392831	392967	393102	393238
24	393373	393509	393644	393779	393914	394050	394185	394320	394455	394590
25	394725	394860	394995	395130	395265	395400	395535	395670	395805	395940
26	396075	396209	396344	396479	396614	396748	396883	397017	397151	397286
27	397421	397555	397689	397824	397958	398093	398227	398361	398495	398630
28	398764	398898	399032	399166	399300	399434	399568	399702	399836	399970
29	400104	400238	400372	400505	400639	400773	400907	401040	401174	401307
30	401441	401574	401708	401842	401975	402108	402242	402375	402509	402642
31	402775	402908	403042	403175	403308	403441	403574	403707	403840	403973
32	404106	404239	404372	404505	404638	404771	404904	405036	405169	405302
33	405435	405567	405700	405832	405964	406097	406230	406362	406495	406627
34	406760	406892	407025	407157	407289	407421	407553	407686	407818	407950
35	408082	408214	408346	408478	408610	408742	408874	409006	409138	409270
36	409401	409533	409665	409797	409928	410060	410192	410323	410455	410586
37	410718	410849	410981	411112	411244	411375	411506	411638	411769	411900
38	412031	412163	412294	412425	412556	412687	412818	412949	413080	413211
39	413342	413473	413604	413734	413865	413996	414127	414258	414388	414519
40	414650	414780	414911	415041	415172	415302	415433	415563	415694	415824
41	415954	416085	416215	416345	416475	416606	416736	416866	416996	417126
42	417256	417386	417516	417646	417776	417906	418036	418166	418295	418425
43	418555	418685	418814	418944	419074	419203	419333	419463	419592	419721
44	419851	419980	420110	420239	420369	420498	420627	420756	420886	421015
45	421144	421273	421402	421531	421660	421790	421919	422048	422177	422306
46	422434	422563	422692	422821	422950	423078	423207	423336	423465	423593
47	423722	423850	423979	424107	424236	424364	424493	424621	424750	424878
48	425006	425134	425263	425391	425519	425647	425775	425904	426032	426160
49	426288	426416	426544	426672	426800	426928	427055	427183	427311	427439
50	427567	427694	427822	427950	428077	428205	428332	428460	428588	428715
51	428843	428970	429097	429225	429352	429479	429607	429734	429861	429988
52	430116	430243	430370	430497	430624	430751	430878	431005	431132	431259
53	431386	431513	431640	431766	431893	432020	432147	432273	432400	432527
54	432653	432780	432906	433033	433159	433286	433412	433539	433665	433792
55	433918	434044	434170	434297	434423	434549	434675	434801	434928	435054
56	435180	435306	435432	435558	435684	435809	435935	436061	436187	436313
57	436439	436564	436670	436816	436941	437067	437193	437318	437444	437569
58	437695	437820	437946	438071	438197	438322	438447	438572	438698	438823
59	438948	439073	439198	439323	439448	439574	439699	439824	439949	440074
60	440199	440323	440448	440573	440698	440823	440948	441072	441197	441322
61	441446	441571	441696	441820	441945	442069	442194	442318	442443	442567
62	442691	442816	442940	443064	443189	443313	443437	443561	443685	443809
63	443933	444058	444182	444306	444430	444554	444678	444801	444925	445049
64	445173	445297	445420	445544	445668	445792	445915	446039	446163	446286
65	446410	446533	446657	446780	446904	447027	447150	447274	447397	447520
66	447644	447767	447890	448013	448136	448260	448383	448506	448629	448752
67	448875	448998	449121	449244	449367	449389	449612	449735	449858	449981
68	450103	450226	450349	450471	450594	450716	450839	450962	451084	451207
69	451329	451451	451574	451696	451818	451941	452063	452175	452308	452430

TABLE 9.—*Values of* $1-R^x$ *when* $R=0.8$—Continued

(All values are decimal fractions)

x	0	1	2	3	4	5	6	7	8	9
2.70	452552	452674	452796	452918	453040	453162	453284	453406	453528	453650
71	453772	453894	454016	454138	454259	454381	454503	454625	454746	454868
72	454990	455111	455233	455354	455476	455597	455719	455840	455962	456083
73	456204	456326	456447	456568	456690	456811	456932	457053	457174	457295
74	457416	457538	457659	457780	457901	458022	458142	458263	458384	458505
75	458626	458747	458867	458988	459109	459230	459350	459471	459592	459712
76	459833	459953	460074	460194	460315	460435	460555	460676	460796	460916
77	461037	461157	461277	461397	461517	461638	461758	461878	461998	462118
78	462238	462358	462478	462598	462718	462838	462958	463077	463197	463317
79	463437	463556	463676	463795	463915	464035	464155	464274	464394	464513
80	464633	464752	464871	464991	465110	465230	465349	465468	465587	465707
81	465826	465945	466064	466183	466302	466422	466541	466660	466779	466898
82	467017	467135	467254	467373	467492	467611	467730	467848	467967	468086
83	468205	468323	468442	468560	468679	468798	468916	469035	469153	469272
84	469390	469508	469627	469745	469863	469981	470100	470218	470336	470454
85	470573	470691	470809	470927	471045	471163	471281	471399	471517	471635
86	471753	471871	471988	472106	472224	472342	472460	472577	472695	472812
87	472930	473048	473165	473283	473400	473518	473635	473753	473870	473988
88	474105	474222	474339	474457	474574	474681	474808	474926	475043	475160
89	475277	475394	475511	475628	475745	475862	475979	476096	476213	476330
90	476447	476563	476680	476797	476914	477030	477147	477264	477380	477497
91	477614	477730	477847	477963	478080	478196	478312	478429	478545	478662
92	478778	478894	479011	479127	479243	479359	479475	479592	479708	479824
93	479940	480056	480172	480288	480404	480520	480636	480751	480867	480983
94	481099	481215	481331	481446	481562	481678	481793	481909	482025	482140
95	482256	482371	482487	482602	482717	482833	482948	483064	483179	483294
96	483410	483525	483640	483755	483871	483986	484101	484216	484331	484446
97	484561	484676	484791	484906	485021	485136	485251	485365	485480	485595
98	485710	485825	485939	486054	486169	486284	486398	486513	486627	486742
99	486856	486971	487085	487200	487314	487428	487543	487657	487772	487886
3.00	488000	488114	489228	488343	488457	488571	488685	488799	488913	489027
01	489141	489255	489369	489483	489597	489711	489825	489939	490052	490166
02	490280	490394	490507	490621	490735	490848	490961	491075	491189	491303
03	491416	491530	491643	491757	491870	491983	492096	492210	492323	492436
04	492550	492663	492776	492889	493002	493115	493229	493342	493455	493568
04	493681	493794	493907	494020	494132	494245	494358	494471	494584	494697
06	494809	494922	495035	495147	495260	495373	495485	495598	495710	495823
07	495935	496048	496160	496273	496385	496497	496610	496722	496834	496947
08	497059	497171	497283	497395	497508	497620	497732	497844	497956	498068
09	498180	498292	498404	498516	498628	498740	498851	498963	499075	499187
10	499298	499410	499522	499633	499745	499857	499968	500080	500191	500303
11	500414	500526	500637	500749	500860	500971	501083	501194	501305	501417
12	501528	501639	501750	501862	501973	502084	502195	502306	502417	502528
13	502639	502750	502861	502972	503083	503194	503305	503415	503526	503637
14	503748	503858	503969	504080	504190	504301	504412	504522	504633	504743
15	504854	504964	505075	505185	505296	505406	505516	505627	505737	505847
16	505958	506068	506178	506288	506398	506508	506618	506729	506839	506949
17	507059	507169	507279	507389	507498	507608	507718	507828	507938	508048
18	508157	508267	508377	508486	508596	508706	508815	508925	509035	509144
19	509254	509363	509473	509582	509691	509801	509910	510020	510129	510238
20	510348	510457	510566	510675	510784	510893	511003	511112	511221	511330
21	511439	511548	511657	511766	511875	511984	512093	512202	512310	512419
22	512528	512637	512745	512854	512963	513072	513180	513289	513397	513506
23	513615	513723	513831	513940	514048	514157	514265	514374	514482	514590
24	514699	514807	514915	515024	515132	515240	515348	515456	515564	515672
25	515780	515888	515996	516104	516212	516320	516428	516536	516644	516752
26	516860	516967	517075	517183	517291	517398	517506	517614	517721	517829
27	517936	518044	518152	518259	518367	518474	518582	518689	518796	518904
28	519011	519118	519226	519333	519440	519547	519654	519762	519869	519976
29	520083	520190	520297	520404	520511	520618	520725	520832	520939	521046
30	521153	521260	521366	521473	521580	521687	521794	521900	522007	522114
31	522220	522327	522433	522540	522646	522753	522859	522966	523072	523178
32	523285	523392	523498	523604	523710	523817	523923	524029	524135	524242
33	524348	524454	524560	524666	524772	524878	524984	525090	525196	525302
34	525408	525514	525620	525725	525831	525937	526043	526148	526254	526360
35	526466	526572	526677	526783	526888	526994	527099	527205	527310	527416
36	527521	527627	527732	527837	527943	528048	528154	528259	528364	528469
37	528574	528680	528785	528890	528995	529100	529205	529310	529415	529520
38	529625	529730	529835	529940	530045	530150	530255	530359	530464	530569
39	530674	530778	530883	530987	531092	531197	531302	531406	531511	531615
40	531720	531824	531929	532033	532137	532242	532346	532451	532555	532659
41	532763	532868	532972	533076	533180	533284	533389	533493	533597	533701
42	533805	533909	534013	534117	534221	534325	534429	534532	534636	534740
43	534844	534948	535052	535155	535259	535363	535466	535570	535674	535777
44	535881	535984	536088	536191	536295	536398	536502	536605	536709	536812
45	536915	537019	537122	537225	537328	537432	537535	537638	537741	537844
46	537947	538050	538154	538257	538360	538463	538566	538669	538772	538874
47	538977	539080	539183	539286	539389	539491	539594	539697	539800	539902
48	540005	540108	540210	540313	540415	540518	540620	540723	540826	540928
49	541030	541133	541235	541337	541440	541542	541644	541747	541849	541951
50	542053	542155	542258	542360	542462	542564	542666	542768	542870	542972
51	543074	543176	543278	543380	543482	543584	543685	543787	543889	543991
52	544092	544194	544296	544398	544499	544601	544702	544804	544906	545007
53	545109	545210	545312	545413	545515	545616	545717	545819	545920	546021
54	546123	546224	546325	546426	546528	546629	546730	546831	546932	547033
55	547134	547235	547336	547437	547538	547639	547740	547841	547942	548043
56	548144	548245	548345	548446	548547	548648	548748	548849	548950	549050
57	549151	549251	549352	549453	549553	549654	549754	549855	549955	550056
58	550156	550256	550357	550457	550557	550657	550758	550858	550958	551058
59	551158	551259	551359	551459	551559	551659	551759	551859	551959	552059

TABLE 9.—*Values of* $1-R^x$ *when* $R=0.8$—Continued

(All values are decimal fractions)

x	0	1	2	3	4	5	6	7	8	9
3.60	552159	552259	552359	552459	552558	552658	552758	552858	552958	553057
61	553157	553257	553356	553456	553556	553655	553755	553855	553954	554054
62	554153	554253	554352	554451	554551	554650	554750	554849	554948	555047
63	555147	555246	555345	555445	555544	555643	555742	555841	555940	556039
64	556138	556237	556336	556435	556534	556633	556732	556831	556930	557029
65	557128	557227	557325	557424	557523	557622	557720	557819	557917	558016
66	558115	558214	558313	558411	558509	558608	558706	558805	558903	559001
67	559100	559198	559297	559395	559493	559591	559690	559788	559886	559984
68	560082	560181	560279	560377	560475	560573	560671	560769	560867	560965
69	561063	561161	561259	561357	561455	561553	561650	561748	561846	561944
70	562041	562139	562237	562335	562432	562530	562627	562725	562823	562920
71	563018	563115	563213	563310	563408	563505	563602	563700	563797	563894
72	563992	564089	564186	564284	564381	564478	564575	564672	564769	564866
73	564964	565061	565158	565255	565352	565449	565546	565643	565739	565836
74	565933	566030	566127	566224	566320	566417	566514	566611	566707	566804
75	566901	566997	567094	567191	567287	567384	567480	567577	567673	567770
76	567866	567963	568059	568155	568252	568348	568444	568541	568637	568733
77	568829	568925	569022	569118	569214	569310	569406	569502	569598	569694
78	569790	569886	569982	570078	570174	570270	570366	570462	570558	570654
79	570749	570845	570941	571037	571132	571228	571324	571419	571515	571610
80	571706	571802	571897	571993	572088	572184	572279	572374	572470	572565
81	572661	572756	572851	572947	573042	573137	573232	573328	573423	573518
82	573613	573708	573803	573899	573994	574089	574184	574279	574374	574469
83	574564	574659	574753	574848	574943	575038	575133	575228	575322	575417
84	575512	575607	575701	575796	575891	575985	576080	576174	576269	576363
85	576458	576552	576647	576741	576836	576931	577025	577119	577213	577308
86	577402	577496	577591	577685	577779	577873	577967	578062	578156	578250
87	578344	578438	578532	578626	578720	578814	578908	579002	579096	579190
88	579284	579378	579472	579565	579659	579753	579847	579941	580034	580128
89	580222	580315	580409	580502	580596	580690	580783	580877	580970	581064
90	581157	581251	581344	581438	581531	581624	581718	581811	581904	581998
91	582091	582184	582277	582370	582464	582557	582650	582743	582836	582929
92	583022	583115	583208	583301	583394	583487	583580	583673	583766	583859
93	583952	584045	584137	584230	584323	584416	584508	584601	584694	584787
94	584879	584972	585064	585157	585249	585342	585435	585527	585619	585712
95	585804	585897	585989	586082	586174	586266	586359	586451	586543	586635
96	586728	586820	586912	587004	587096	587189	587281	587373	587465	587557
97	587649	587741	587833	587925	588017	588109	588201	588292	588384	588476
98	588568	588660	588751	588843	588935	589027	589118	589210	589302	589393
99	589485	589577	589668	589760	589851	589943	590034	590126	590217	590309
4.0	59040	59131	59222	59313	59404	59494	59585	59675	59765	59854
1	59944	60033	60122	60211	60300	60388	60477	60565	60653	60740
2	60828	60915	61002	61089	61176	61262	61349	61435	61521	61607
3	61692	61778	61863	61948	62033	62117	62202	62286	62370	62454
4	62538	62621	62704	62787	62870	62953	63036	63118	63200	63282
5	63364	63446	63527	63609	63690	63771	63851	63932	64012	64093
6	64173	64253	64332	64412	64491	64570	64649	64728	64807	64885
7	64963	65041	65119	65197	65275	65352	65429	65506	65583	65660
8	65736	65813	65889	65965	66041	66117	66192	66268	66343	66418
9	66493	66567	66642	66716	66790	66864	66938	67012	67085	67159
5.0	67232	67305	67378	67451	67523	67596	67668	67740	67812	67883
1	67955	68027	68098	68169	68240	68311	68381	68451	68522	68592
2	68662	68732	68802	68871	68941	69010	69079	69148	69217	69285
3	69354	69422	69490	69558	69626	69694	69761	69829	69896	69963
4	70030	70097	70164	70230	70296	70363	70429	70495	70560	70626
5	70691	70757	70822	70887	70952	71016	71081	71146	71210	71274
6	71338	71402	71466	71529	71592	71656	71719	71782	71845	71908
7	71971	72033	72095	72158	72220	72282	72344	72406	72467	72528
8	72589	72650	72711	72772	72833	72893	72954	73014	73074	73134
9	73194	73254	73313	73373	73432	73491	73551	73610	73668	73727
6.0	73786	73844	73902	73960	74018	74076	74134	74192	74249	74307
1	74364	74421	74478	74535	74592	74649	74705	74761	74818	74874
2	74930	74986	75041	75097	75153	75208	75263	75318	75373	75428
3	75483	75538	75592	75647	75701	75755	75809	75863	75917	75970
4	76024	76077	76131	76184	76237	76290	76343	76396	76448	76501
5	76553	76605	76658	76710	76761	76813	76865	76917	76968	77019
6	77071	77122	77173	77224	77274	77325	77375	77426	77476	77526
7	77576	77626	77676	77726	77776	77825	77875	77924	77973	78022
8	78071	78120	78169	78218	78266	78315	78363	78411	78459	78507
9	78555	78603	78651	78698	78746	78793	78840	78888	78935	78982
7.0	79028	79075	79122	79168	79215	79261	79307	79354	79400	79445
1	79491	79537	79583	79628	79674	79719	79764	79809	79854	79899
2	79944	79989	80033	80078	80122	80166	80211	80255	80299	80343
3	80386	80430	80474	80517	80561	80604	80647	80690	80733	80776
4	80819	80862	80905	80947	80990	81032	81074	81117	81159	81201
5	81242	81284	81326	81367	81409	81451	81492	81533	81574	81615
6	81656	81697	81738	81779	81819	81860	81900	81941	81981	82021
7	82061	82101	82141	82181	82221	82260	82300	82339	82379	82418
8	82457	82496	82535	82574	82613	82652	82690	82729	82768	82806
9	82844	82882	82920	82959	82997	83035	83072	83110	83148	83185
8.0	83223	83260	83297	83335	83372	83409	83446	83483	83520	83557
1	83593	83630	83666	83702	83739	83775	83811	83847	83883	83919
2	83955	83991	84027	84062	84098	84133	84168	84204	84239	84274
3	84309	84344	84379	84414	84449	84483	84518	84552	84587	84621
4	84655	84690	84724	84758	84792	84826	84859	84893	84927	84960
5	84994	85027	85061	85094	85127	85161	85194	85227	85259	85292
6	85325	85358	85391	85423	85456	85488	85520	85552	85585	85617
7	85649	85681	85713	85745	85776	85808	85840	85872	85903	85934
8	85966	85997	86028	86059	86090	86121	86152	86183	86214	86245
9	86274	86305	86336	86367	86397	86428	86458	86488	86518	86548

TABLE 9.—*Values of* $1-R^x$ *when* $R=0.8$—Continued

(All values are decimal fractions)

x	0	1	2	3	4	5	6	7	8	9
9.0	86578	86608	86638	86668	86697	86727	86757	86786	86816	86845
1	86875	86904	86933	86962	86991	87020	87049	87078	87107	87136
2	87164	87193	87221	87250	87278	87306	87335	87363	87391	87419
3	87447	87475	87503	87531	87559	87587	87614	87642	87669	87697
4	87724	87752	87779	87806	87833	87860	87887	87915	87942	87968
5	87995	88022	88049	88075	88102	88128	88155	88181	88208	88234
6	88260	88286	88312	88338	88364	88390	88416	88442	88468	88494
7	88519	88545	88570	88596	88621	88647	88672	88697	88722	88747
8	88773	88798	88823	88847	88872	88897	88922	88947	88971	88996
9	89020	89045	89069	89094	89118	89142	89166	89190	89215	89239
10.0	89263	89287	89310	89334	89358	89382	89405	89429	89453	89476
1	89500	89523	89546	89570	89593	89616	89639	89662	89685	89708
2	89731	89754	89777	89800	89822	89845	89868	89890	89913	89936
3	89958	89980	90003	90025	90047	90069	90091	90113	90135	90157
4	90179	90201	90223	90245	90267	90288	90310	90332	90353	90375
5	90396	90418	90439	90460	90482	90503	90524	90545	90566	90587
6	90608	90629	90650	90671	90692	90712	90733	90754	90774	90795
7	90815	90836	90856	90877	90897	90917	90937	90958	90978	90998
8	91018	91038	91058	91078	91098	91118	91138	91157	91177	91197
9	91216	91236	91255	91275	91294	91314	91333	91352	91372	91391
11.0	91410	91429	91448	91467	91486	91505	91524	91543	91562	91581
1	91600	91618	91637	91656	91674	91693	91711	91730	91748	91767
2	91785	91803	91822	91840	91858	91876	91894	91912	91930	91948
3	91966	91984	92002	92020	92038	92055	92073	92091	92108	92126
4	92144	92161	92179	92196	92213	92231	92248	92265	92283	92300
5	92317	92334	92351	92368	92385	92402	92419	92436	92453	92470
6	92487	92503	92520	92537	92553	92570	92586	92603	92619	92636
7	92652	92669	92685	92701	92718	92734	92750	92766	92782	92798
8	92814	92830	92846	92862	92878	92894	92910	92926	92942	92957
9	92973	92989	93004	93020	93035	93051	93066	93082	93097	93113
12.0	93128	93143	93159	93174	93189	93204	93219	93235	93250	93265
1	93280	93295	93310	93324	93339	93354	93369	93384	93399	93413
2	93428	93443	93457	93472	93486	93501	93515	93530	93544	93559
3	93573	93587	93602	93616	93630	93644	93658	93673	93687	93701
4	93715	93729	93743	93757	93771	93785	93798	93812	93826	93840
5	93853	93867	93881	93895	93908	93922	93935	93949	93962	93976
6	93989	94003	94016	94029	94043	94056	94069	94082	94095	94109
7	94122	94135	94148	94161	94174	94187	94200	94213	94226	94239
8	94252	94264	94277	94290	94303	94315	94328	94341	94353	94366
9	94378	94391	94403	94416	94428	94441	94453	94465	94478	94490
13.0	94502	94515	94527	94539	94551	94563	94575	94588	94600	94612
1	94624	94636	94648	94660	94671	94683	94695	94707	94719	94731
2	94742	94754	94766	94777	94789	94801	94812	94824	94835	94847
3	94858	94870	94881	94893	94904	94915	94927	94938	94949	94961
4	94972	94983	94994	95005	95017	95028	95039	95050	95061	95072
5	95083	95094	95105	95116	95126	95137	95148	95159	95170	95181
6	95191	95202	95213	95223	95234	95245	95255	95266	95276	95287
7	95297	95308	95318	95329	95339	95349	95360	95370	95381	95391
8	95401	95411	95422	95432	95442	95452	95462	95472	95483	95493
9	95503	95513	95523	95533	95543	95553	95563	95572	95582	95592
14.0	95602	95612	95621	95631	95641	95651	95660	95670	95680	95689
1	95699	95709	95718	95728	95737	95747	95756	95766	95775	95784
2	95794	95803	95813	95822	95831	95841	95850	95859	95868	95878
3	95887	95896	95905	95914	95923	95932	95941	95951	95960	95969
4	95978	95987	95995	96004	96013	96022	96031	96040	96049	96058
5	96066	96075	96084	96092	96101	96110	96119	96127	96136	96144
6	96153	96162	96170	96179	96187	96196	96204	96213	96221	96230
7	96238	96246	96255	96263	96271	96280	96288	96296	96304	96313
8	96321	96329	96337	96346	96354	96362	96370	96378	96386	96394
9	96402	96410	96418	96426	96434	96442	96450	96458	96466	96474
15.0	96482	96489	96497	96505	96513	96521	96528	96536	96544	96551
1	96559	96567	96574	96582	96590	96597	96605	96612	96620	96628
2	96635	96643	96650	96658	96665	96672	96680	96687	96695	96702
3	96709	96717	96724	96731	96739	96746	96753	96760	96768	96775
4	96782	96789	96796	96803	96811	96818	96825	96832	96839	96846
5	96853	96860	96867	96874	96881	96888	96895	96902	96909	96916
6	96923	96929	96936	96943	96950	96957	96963	96970	96977	96984
7	96991	96998	97004	97010	97017	97024	97030	97037	97044	97050
8	97057	97063	97070	97076	97083	97089	97096	97102	97109	97115
9	97122	97128	97135	97141	97147	97154	97160	97166	97173	97179
16.0	97185	97192	97198	97204	97210	97216	97223	97229	97235	97241
1	97247	97253	97260	97266	97272	97278	97284	97290	97296	97302
2	97308	97314	97320	97326	97332	97338	97344	97350	97356	97362
3	97368	97373	97379	97385	97391	97397	97403	97408	97414	97420
4	97426	97431	97437	97443	97448	97454	97460	97465	97471	97477
5	97482	97488	97494	97499	97505	97510	97516	97521	97527	97532
6	97538	97544	97549	97554	97560	97565	97571	97576	97581	97587
7	97592	97598	97603	97608	97614	97619	97624	97630	97635	97640
8	97645	97651	97656	97661	97666	97671	97677	97682	97687	97692
9	97697	97702	97708	97713	97718	97723	97728	97733	97738	97743
17.0	97748	97753	97758	97763	97768	97773	97778	97783	97788	97793
1	97798	97803	97808	97813	97817	97822	97827	97832	97837	97842
2	97846	97851	97856	97861	97866	97870	97875	97880	97885	97889
3	97894	97899	97903	97908	97913	97917	97922	97927	97931	97936
4	97940	97945	97950	97954	97959	97963	97968	97972	97977	97981
5	97986	97990	97995	97999	98004	98008	98013	98017	98022	98026
6	98030	98035	98039	98043	98048	98052	98056	98061	98065	98069
7	98074	98078	98082	98087	98091	98095	98099	98104	98108	98112
8	98116	98120	98125	98129	98133	98137	98141	98146	98150	98154
9	98158	98162	98166	98170	98174	98178	98182	98186	98190	98194

TABLE 9.—*Values of $1-R^x$ when $R=0.8$*—Continued

(All values are decimal fractions)

x	0	1	2	3	4	5	6	7	8	9
18.0	98199	98203	98207	98211	98215	98219	98223	98226	98230	98234
1	98238	98242	98246	98250	98254	98258	98262	98266	98269	98273
2	98277	98281	98285	98289	98292	98296	98300	98304	98308	98311
3	98315	98319	98323	98326	98330	98334	98338	98341	98345	98349
4	98352	98356	98360	98363	98367	98371	98374	98378	98382	98385
5	98389	98392	98396	98399	98403	98406	98410	98414	98417	98421
6	98424	98428	98431	98435	98438	98442	98445	98449	98452	98456
7	98459	98462	98466	98469	98472	98476	98480	98483	98486	98490
8	98494	98497	98500	98503	98506	98510	98513	98516	98520	98523
9	98526	98530	98533	98536	98539	98543	98546	98549	98552	98556
19.0	98559	98562	98565	98568	98572	98575	98578	98581	98584	98587
1	98591	98594	98597	98600	98603	98606	98609	98612	98616	98619
2	98622	98625	98628	98631	98634	98637	98640	98643	98646	98649
3	98652	98655	98658	98661	98664	98667	98670	98673	98676	98679
4	98682	98685	98688	98691	98694	98697	98699	98702	98705	98708
5	98711	98714	98717	98720	98722	98725	98728	98731	98734	98737
6	98739	98742	98745	98748	98751	98753	98756	98759	98762	98765
7	98767	98770	98773	98775	98778	98781	98784	98786	98789	98792
8	98794	98797	98800	98802	98805	98808	98810	98813	98816	98819
9	98821	98824	98826	98829	98832	98834	98837	98839	98842	98845
20.0	98847	98850	98852	98855	98857	98860	98862	98865	98867	98870
1	98873	98876	98878	98880	98883	98885	98888	98890	98892	98895
2	98897	98900	98902	98905	98907	98910	98912	98914	98917	98919
3	98922	98924	98927	98929	98931	98934	98936	98938	98941	98943
4	98945	98948	98950	98953	98955	98957	98960	98962	98964	98966
5	98969	98971	98973	98976	98978	98980	98982	98985	98988	98990
6	98992	98994	98996	98998	99001	99003	99005	99007	99009	99012
7	99014	99016	99018	99020	99023	99025	99027	99029	99031	99034
8	99036	99038	99040	99042	99044	99046	99048	99051	99053	99055
9	99057	99059	99061	99063	99065	99067	99069	99071	99073	99075
21.0	99077	99080	99082	99084	99086	99088	99090	99092	99094	99096
1	99098	99100	99102	99104	99106	99108	99110	99112	99114	99116
2	99118	99120	99122	99124	99126	99128	99130	99132	99134	99136
3	99137	99139	99141	99143	99145	99147	99149	99151	99153	99155
4	99157	99158	99160	99162	99164	99166	99168	99170	99171	99173
5	99175	99177	99179	99181	99182	99184	99186	99188	99190	99192
6	99193	99195	99197	99199	99201	99202	99204	99206	99208	99209
7	99211	99213	99215	99216	99218	99220	99222	99223	99225	99227
8	99229	99230	99232	99234	99235	99237	99239	99241	99242	99244
9	99246	99247	99249	99251	99252	99254	99256	99257	99259	99261
22.0	99262	99264	99266	99267	99269	99270	99272	99274	99275	99277
1	99279	99280	99282	99283	99285	99286	99288	99290	99291	99293
2	99294	99296	99298	99299	99300	99302	99304	99305	99307	99308
3	99310	99312	99313	99315	99316	99318	99319	99321	99322	99324
4	99325	99327	99328	99330	99331	99333	99334	99336	99338	99339
5	99340	99341	99343	99344	99346	99347	99349	99350	99352	99353
6	99355	99356	99358	99359	99360	99362	99363	99365	99366	99368
7	99369	99370	99372	99373	99374	99376	99377	99379	99380	99381
8	99383	99384	99386	99387	99388	99390	99391	99392	99394	99395
9	99396	99398	99399	99401	99402	99403	99405	99406	99407	99408
23.0	99410	99411	99412	99414	99415	99416	99418	99419	99420	99421
1	99423	99424	99425	99427	99428	99429	99431	99432	99433	99434
2	99436	99437	99438	99439	99440	99442	99443	99445	99446	99447
3	99448	99449	99450	99452	99453	99454	99455	99456	99458	99459
4	99460	99461	99463	99464	99465	99466	99467	99468	99470	99471
5	99472	99473	99474	99476	99477	99478	99479	99480	99481	99483
6	99484	99485	99486	99487	99488	99490	99491	99492	99493	99494
7	99495	99496	99497	99498	99500	99501	99502	99503	99504	99505
8	99506	99507	99508	99509	99511	99512	99513	99514	99515	99516
9	99517	99518	99519	99520	99521	99522	99524	99525	99526	99527
24.0	99528	99529	99530	99531	99532	99533	99534	99535	99536	99537
1	99538	99539	99540	99541	99542	99543	99544	99545	99546	99547
2	99548	99549	99550	99551	99552	99553	99554	99555	99556	99557
3	99558	99559	99560	99561	99562	99563	99564	99565	99566	99567
4	99568	99569	99570	99571	99572	99573	99574	99575	99576	99577
5	99578	99579	99580	99581	99581	99582	99583	99584	99585	99586
6	99587	99588	99589	99590	99591	99592	99593	99593	99594	99595
7	99596	99597	99598	99599	99600	99601	99602	99602	99603	99604
8	99605	99606	99607	99608	99609	99609	99610	99611	99612	99613
9	99614	99615	99615	99616	99617	99618	99619	99620	99621	99621
25.0	99622	99623	99624	99625	99626	99626	99627	99628	99628	99629
1	99630	99631	99632	99633	99634	99635	99636	99636	99637	99638
2	99639	99640	99640	99641	99642	99643	99644	99644	99645	99646
3	99647	99648	99648	99649	99650	99651	99651	99652	99653	99654
4	99655	99655	99656	99657	99658	99659	99659	99660	99661	99661
5	99662	99663	99664	99664	99665	99666	99667	99668	99668	99669
6	99670	99670	99671	99672	99672	99673	99674	99675	99675	99676
7	99677	99678	99678	99679	99680	99681	99681	99682	99683	99683
8	99684	99685	99685	99686	99687	99687	99688	99689	99690	99690
9	99691	99692	99692	99693	99694	99694	99695	99696	99696	99697
26.0	99698	99699	99699	99700	99700	99701	99702	99703	99703	99704
1	99704	99705	99706	99706	99707	99707	99708	99708	99709	99710
2	99711	99712	99712	99713	99714	99714	99715	99716	99716	99717
3	99717	99718	99719	99719	99720	99720	99721	99722	99722	99723
4	99724	99724	99725	99726	99726	99727	99727	99728	99728	99729
5	99730	99730	99731	99731	99732	99733	99733	99734	99734	99735
6	99736	99736	99737	99738	99738	99739	99739	99740	99740	99741
7	99741	99742	99743	99743	99744	99744	99745	99746	99746	99747
8	99747	99748	99749	99749	99750	99750	99751	99751	99752	99752
9	99753	99753	99754	99755	99755	99756	99756	99757	99757	99758

TABLE 9.—*Values of* $1-R^x$ *when* $R=0.8$—Continued

(All values are decimal fractions)

x	0	1	2	3	4	5	6	7	8	9
27.0	99759	99759	99759	99760	99760	99761	99761	99762	99762	99763
1	99764	99764	99765	99765	99766	99766	99767	99767	99768	99768
2	99769	99769	99770	99770	99771	99771	99772	99772	99773	99773
3	99774	99774	99775	99775	99776	99776	99777	99777	99778	99778
4	99779	99779	99780	99780	99781	99781	99782	99782	99783	99783
5	99784	99784	99785	99785	99786	99786	99787	99787	99788	99788
6	99789	99789	99789	99790	99790	99791	99791	99792	99792	99793
7	99793	99794	99794	99795	99795	99796	99796	99796	99797	99797
8	99798	99798	99799	99799	99800	99800	99800	99801	99801	99802
9	99802	99803	99803	99804	99804	99805	99805	99805	99806	99806
28.0	99807	99807	99807	99808	99808	99809	99809	99810	99810	99810
1	99811	99811	99812	99812	99813	99813	99813	99814	99814	99815
2	99815	99816	99816	99816	99817	99817	99817	99818	99818	99819
3	99819	99820	99820	99820	99821	99821	99821	99822	99822	99823
4	99823	99823	99824	99824	99825	99825	99825	99826	99826	99827
5	99827	99827	99828	99828	99829	99829	99829	99830	99830	99830
6	99831	99831	99832	99832	99832	99833	99833	99833	99834	99834
7	99835	99835	99835	99836	99836	99836	99836	99837	99837	99838
8	99838	99839	99839	99839	99840	99840	99840	99841	99841	99841
9	99842	99842	99842	99843	99843	99844	99844	99844	99845	99845
29.0	99845	99846	99846	99846	99847	99847	99847	99848	99848	99848
1	99849	99849	99849	99850	99850	99850	99851	99851	99851	99852
2	99852	99852	99853	99853	99853	99854	99854	99854	99855	99855
3	99855	99856	99856	99856	99857	99857	99857	99858	99858	99858
4	99858	99859	99859	99859	99860	99860	99860	99861	99861	99861
5	99862	99862	99862	99862	99863	99863	99863	99864	99864	99864
6	99865	99865	99865	99866	99866	99866	99866	99867	99867	99867
7	99868	99868	99868	99869	99869	99869	99869	99870	99870	99870
8	99871	99871	99871	99871	99872	99872	99872	99873	99873	99873
9	99873	99874	99874	99874	99875	99875	99875	99875	99876	99876
30	998762	---	---	---	---	---	---	---	---	---
31	999010	---	---	---	---	---	---	---	---	---
32	999208	---	---	---	---	---	---	---	---	---
33	999366	---	---	---	---	---	---	---	---	---
34	999493	---	---	---	---	---	---	---	---	---
35	999594	---	---	---	---	---	---	---	---	---
36	999675	---	---	---	---	---	---	---	---	---
37	999740	---	---	---	---	---	---	---	---	---
38	999792	---	---	---	---	---	---	---	---	---
39	999834	---	---	---	---	---	---	---	---	---
40	999866	---	---	---	---	---	---	---	---	---

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